Structured Life Insurance and Investment Products for Retail Investors

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Chapter 1

Introduction and Motivation

" I wish someone would give me one shred of neutral evidence that financial innovation has let to economic growth, one shred of evidence."
Paul Volcker

Structured life insurance and investment products combine individual financial instruments such as bonds and stocks with positions in financial derivatives. These products are tailored to give retail investors the opportunity to optimize their investment portfolios by including derivative structures and strategies which are usually not available to retail investors. How the optimal portfolio looks like depends on the motives of the retail investor. Nowadays, the intention behind the portfolio decisions of private households is influenced to a great extend by the demographic change which has been observed in the last twenty years. Decreasing birth rates combined with increasing life expectancy cause a gap in the national old-age pension system. Thus, to fill this gap the investor has additionally to rely on private pension schemes and/or other investments which are also supported by government incentives for savers. Consequently, the investment decision crucially influences the investor's situation at retirement. Thereby, the investor has to decide on the trade–off between security necessary to guarantee the desired minimum standard of living at retirement and seeking for high profits. Structured life insurance and investment products are especially tailored to suit the retail investors needs and expectations towards an investment. The question arises whether the offered products really fulfill what they claim. This thesis tries to approach this question.

The first part considers structured life insurance contracts. Here, the first question is how do traded structured life insurance products look like. This question is answered in Chapter 2 where an overview of product characteristics and the academic literature on these products is provided. It is highlighted that due to the flexibility in the design of this products, the interaction between insurance company and insurance taker becomes of great importance. Therefore, this chapter also motivates the two following chapters. Chapter 3 answers the question which structured life insurance product is indeed optimal for an retail investor. In particular, Chapter 3 considers structured life insurance contracts where the benefits of the insured depend on the performance of an investment strategy and which guarantee a certain interest rate on the contributions made by the insured. The insured has to decide simultaneously on the investment strategy and the guarantee scheme, i.e. the particular structure of the product. In particular, we consider the so-called contribution guaranteed scheme which is designed as guaranteed minimum accumulation benefits belonging to the class of variable annuities and the participation surplus scheme which resembles equity-linked life insurance contracts. We analyze optimal contracts in the sense of utility maximization. We formulate the overall maximization problem restricting the insured to the two guarantee schemes but not the class of the investment strategy except the self-financing condition. For a CRRA utility investor and in a Black-Scholes economy, the optimal combination is given by a constant mix strategy underlying the contribution guarantee scheme. In case the insured has a subsistence level, the Constant Proportional Portfolio Insurance (CPPI) strategy turns out to be optimal for arbitrary schemes. We illustrate our results by numerical examples and analyze the utility losses of a CRRA insured due to the use of a suboptimal combination of investment strategy and guarantee scheme. Both the exogenous guarantee and the restriction to a fixed set of contracts lead to utility losses for the insured. We show that the losses due to the guarantee by far exceed the losses due to the use of a suboptimal investment strategy or guarantee scheme, in particular for short times to maturity.

Chapter 4 focuses on an additional rider included in guaranteed minimum accumulation benefits. The payoff of these products is linked to the performance of a multi-asset investment strategy and includes a minimum interest rate guarantee on the contributions. In addition, the buyer receives the option to decide on the investments dynamically. Due to the embedded guarantee, these products are interesting for risk averse investors who, in general, benefit from diversification. However, to stay on the safe side, the price setting of the provider must take into account the most risky strategy. We show that this implies an incentive to invest more riskily than without the additional rider. In particular, we quantify the trade-off between the utility of diversification and the utility of a more valuable guarantee relying on realistic examples. In addition, it turns out that a product design including the additional flexibility on the investment decisions causes significant utility losses. The analysis is extended to the situation where the insured receives additional non-market wealth. Possible sources of this background asset are, e.g. retirement income (public pension scheme), real estate, or bequests. Qualitatively, the results without background asset do not change introducing the non-market wealth.

The second part of this thesis considers structured investment products. In a first step one particular structured investment product a so-called relax certificate is analyzed. The offered certificates have become increasingly more complex in recent years and therefore harder to understand for retail investors. Here, the question arises whether the promised features of a relax certificate are really that appealing for retail investors. Moreover, empirical studies show that certificates are often significantly overpriced. Furthermore, Stoimenov and Wilkens (2005) find that the overpricing is the larger the more complex the product is. The payoff of a relax certificate depends on a barrier condition such that it is path-dependent. As long as none of the underlying assets crosses a lower barrier, the investor receives the payoff of a coupon bond. Otherwise, there is a cash settlement at maturity which depends on the lowest stock return. Thus, the product consists of a knock-out coupon bond and a knock-in minimum option. In a Black-Scholes model setup, the price of the knock-out part can be given in closed (or semi-closed) form in the case of one or two underlyings, but not for more than two. In this thesis tight and tractable upper bounds for the great majority of traded products are derived. Comparing with market data it comes out that relax certificates are significantly overpriced. There are two possible conclusions. First, relax certificates may be overpriced in the market. The mispricing is the higher the higher the bonus payments (and thus the higher the discount due to the knock-out feature of the bond). We conjecture that the investors do not correctly estimate the risk associated with the barrier feature, but overweight the sure coupon. Second, the model of Black-Scholes may not be the appropriate choice. However, we argue that extensions to on average downward jumps or default risk of the issuer would result in even lower prices than our upper price bounds. We thus conclude that it is hard to find a model-based motivation for the large prices of relax certificates at the market and that there is strong evidence that these contracts are indeed overpriced.

The second chapter of Part II considers financial strategies which are designed to limit downside risk and at the same time to profit from rising markets. These strategies are summarized in the class of portfolio insurance strategies. The most prominent examples are CPPI strategies and protective put strategies. In practice, the CPPI strategy is the dominating one and often used in the context of Riester products. As we have seen in the previous chapters the optimality of an investment strategy depends on the risk profile of the investor. Portfolio insurers can be modelled by utility maximizers where the maximization problem is given under the additional constraint that the value of the strategy is above a specified wealth level. Independent of the utility function, the solution of the maximization problem is given by the unconstrained problem including a put option as long as the price dynamics are smooth. Obviously, this is in the spirit of the protective put. However, in the special situation of a HARA investor in a Black-Scholes economy, the optimal strategy can be interpreted as a CPPI which honors the guarantee already by construction. This implies that an additional put option becomes obsolete. In this chapter we analyze situations under which the CPPI strategy can be optimal even for a CRRA investor. For the protective put strategy, the price for the guarantee, i.e. the price of the option, is deducted from the initial investment premium the investor pays at inception of the contract. We argue that due to market conditions, e.g. implied volatility vs. historical volatility, mispriced put options, the put solution does not have to result in an optimal *fair* contract, i.e. the present value of the contributions of the investor does not have to coincide with the present value of the benefits which result from the optimal strategy. In contrast, the CPPI strategy does not require buying an additional instrument. Even a slight deviation from *fair* contract specification can make the CPPI strategy more attractive even for a CRRA investor. In addition, we include borrowing constraints for both strategies. Interestingly, a CPPI is less harmed by the introduction of borrowing constraints, than the put solution.

The remaining part of this thesis is structured as follows: Part I Structured Life Insurance Products contains chapters 2-4. Chapter 2 gives an overview on structured life insurance products and the related literature. Chapter 3 explores the optimal structure and investment strategy of structured life insurance products. Chapter 4 examines the impacts of an additional rider on the investors portfolio decisions. Part II Structured Insurance Products consists of chapter five and six. Chapter **??** prices a currently traded certificate relying on upper bounds. In Chapter 6 the utility of portfolio insurance strategies under mispriced put-options and borrowing constraints is analyzed. Chapter 7 concludes the thesis.

Part I

Structured Life Insurance Products

Chapter 2
Structured Life Insurance Products: A Survey

2.1 Introduction

This chapter gives an overview on the literature dealing with the pricing, risk management and product design of structured life insurance products (SLIPs), i.e. life insurance claims with an inherent financial risk. Their payoff is linked to an underlying risky investment strategy which cannot fall below some guaranteed amount. Additionally, innovative riders, the insured can decide on, reveal a new significance to the interaction between insurance company and policyholder.

2.1.1 From classical life insurance to structured life insurance products

The classical actuarial approach to the valuation of life insurance policies considers only the pure insurance risk, i.e. financial risks are assumed to be deterministic. Concerning the insurance risk two important assumptions are often made in the literature on insurance mathematics. First of all, the insurance risk is supposed to be independent of the financial market risk. This is a reasonable assumption and allows a separate treatment of insurance and financial risk.[1] Second, the insurance company does not charge any risk premium for taking on the insurance risk. In fact, the number of policyholders is usually very large in life insurance portfolios. Thus, for a (sufficiently) large cohort the actual number of survivors can be approximated by the expected num-

[1] For a comparison between the actuarial approach and the financial approach to the valuation of life and pension insurance contracts, see Embrechts (1993).

ber of survivors, which implies that the randomness is perfectly diversified.
Since the early 1950s insurance contracts have been designed where the premiums are invested in a stochastic reference portfolio, e.g. mutual funds or simply a portfolio of stocks. However, the financial risk inherent in these so-called pure equity- or pure unit-linked products is completely transferred to the insured which is an undesirable feature in times of financial turmoil. Therefore, and partly by regulatory requirements,[2] contracts have been launched which provide a minimum return guarantee on the contributions of the policyholder and at the same time enable participation in the market. Thus, these contracts present a combination of a classical life insurance contract and an investment strategy. The policyholder receives guaranteed benefits from the life insurance and participates in the profits generated by the underlying investment.

Products which fall in this definition are unit/equity-linked life insurance contracts (UK + continental Europe), equity-indexed annuities (US), segregated funds (Canada) and variable annuities (US (1955), Japan (1999), Europe (2005), Canada (2007)).[3] The basic difference lies in the deduction of the guarantee fee. The first two contracts provide the insured usually a participation by less than 100% in the gains of the chosen investment strategy. The latter two contracts rely on the investment in separate investment accounts which are backed up by a put option to hedge the guarantee. The costs for the guarantee are deducted by decomposing the total premium into an investment premium and a guarantee premium. We subsume these products under the class of structured life insurance products (SLIP) which allow an investment in risky assets like mutual funds but guarantee a minimum level of wealth at the end of the accumulation period. To asses all insurance and financial market risk inherent in these contracts, it is essential to account for the embedded options and to develop meaningful concepts for pricing and risk management.

[2] In Germany, for instance, government-subsidized pension schemes as well as pension schemes from the second pillar of retirement savings must include a guarantee on the contributions.

[3] According to the German Insurance Association (GDV) the market share of new business for unit-linked pension insurance increased from 15.9 % in 2006 to 23.7% in 2008. In 2004 24% of Variable Annuity (VA) policies in the US included a guaranteed minimum accumulation benefit (GMAB) feature. For an overview on the market development of VAs around the world, see Ledlie et al. (2008).

As recognized early on in the literature, see Brennan and Schwartz (1976) and Boyle and Schwartz (1977), the embedded claims have to be priced by no-arbitrage arguments rather than by the traditional actuarial approach.[4] The valuation principle for the financial market risk is based on duplication. In a complete financial market model, for every contingent claim there exists a self-financing strategy duplicating the final payoff. The initial costs of this strategy correspond to the price of the contingent claim. Alternatively, the arbitrage-free price of a contingent claim can be determined by taking the expectation of the discounted payoff under a specific probability measure: the equivalent martingale measure (see Harrison and Kreps (1979) and Harrison and Pliska (1981)). Following the framework of Brennan and Schwartz (1976) the uncertainty of the insured's individual death, respectively survival, is superseded by the expected values according to the law of large numbers. Then, the generalized principle of equivalence is applied to the combined contract to determine the fair contract.[5] Hence, the valuation of the combined contract is defined by taking the expectations under the equivalent martingale measure, i.e. a portfolio of options weighted with the expected death probabilities. Thus, this kind of contract is exposed to insurance and financial market risk.

2.1.2 Characteristics of structured life insurance products

SLIPs combine the flexibility of pure investments like mutual funds with the security of investment strategies with guaranteed downside protection. SLIPs possess a wide choice of underlying reference portfolios, e.g. indices, baskets of mutual funds, equity, tailored investment portfolios a.s.o.. Besides, additional riders are provided to the insured. These concern the flexibility in the premium payments, the investment decision

[4] The traditional actuarial approach takes expectations under the real world measure (insurance risk is assumed to be diversifiable). According to the fundamental work of Black and Scholes (1973) the pricing of options has to rely on duplication (hedging) arguments, thus expectations are taken with regard to the equivalent martingale measure.

[5] The principle of equivalence states that the premiums are chosen such that the present values of contributions and benefits coincide in expectation.

(i.e. the insured chooses the investment strategy underlying the insurance contract), switching rights (i.e. the investment value can be switched between different mutual funds), early redemption rights (i.e. the contract can be cancelled before maturity) and different kind of minimum return guarantee styles. In Table 2.1 the different features

Structured Insurance Product	Contract Features
Investment	Different Mutual Funds
Premium Payment	Single up-front, periodical, periodical but not regular
Investment Decision	Insurance vs Insured ⇒ Option to switch/shift the account value
Maturity	usually 5-20 years ⇒ Option to surrender
Guarantee Styles	Return of Premium, Roll-Up, Ratchet, Reset, Cliquet

Table 2.1 Product features

The Table summarizes different features provided to policyholders of structured insurance products.

attached to SLIPs are summarized. The guarantee features relate to the calculation of the excess between fund performance and guarantee as well as to the specific calculation of the provided guarantee. Here, slight differences do exist between the various traded products. The roll-up style is characterized by guaranteeing a specified minimum rate of return, i.e. interest rate on the principal invested. If the roll-up rate is equal to zero the policyholder owns a so-called return-of-premium guarantee. In the case of a cliquet style guarantee the guarantee rate is granted periodically on the return of the reference portfolio. Sometimes the ratchet style guarantee is defined in a similar way. However, in some cases the ratchet style guarantee is given in terms of the maximum of the initial predetermined guarantee and the growth in the reference portfolio at stipulated dates. The reset guarantee is similar to a ratchet but the guarantee is adjusted to the value of the reference portfolio at stipulated dates where the decision to exercise this feature is at the insured's discretion. A reset style guarantee often comes along with an additional extension of the time to maturity. To sum up, embedded options in the SLIP are of various nature and are in general not plain vanilla. Furthermore, even if a plain vanilla option is included the underlying is often a complex investment strategy. Therefore pricing and risk management is challenging. The following sections aim to give an overview on the existing literature devoted to the pricing and the risk management of these embedded options.

2.2 Classification of the relevant literature

In contrast to participating life insurance contracts, SLIP contracts are funded by separate accounts - not the insurer's general account. Thus, the insurance company establishes a trust on behalf of the policyholder.[6] The scope of this section is to give a structured overview on the literature dealing with the valuation of the chosen SLIP.

2.2.1 Model choice

First, the existing literature on the chosen SLIP can be classified according to the choice of the stochastic model setup. Here, we differentiate between three categories, the model for the stochastic reference portfolio, the interest rate risk model and the mortality risk model.

2.2.1.1 Basic models

Most of the literature considers a perfect economy context, without any transaction costs, administrative costs, or other frictions which impede Black-Scholes-Merton assumptions. This is in the spirit of the pioneering work by Brennan and Schwarz (1976) who take into account single up-front and discrete periodical premiums. Periodical premiums prevent closed-form solutions to the embedded option as this option resembles a discrete time Asian option. Boyle and Schwartz (1977) extend the analysis to

[6] In participating life insurance contracts equity and liability of the insurance company are explicitly modelled as in Briys and de Varenne (1997) and Grosen and Jørgensen (2002) where the insured participates in the general account of the insurance company. For participating life insurance contracts we refer to Briys and de Varenne (2001) and the literature overview therein.

the case of continuous periodical premiums where the embedded option remains an Asian option. For pricing the Asian option, the authors use numerical methods such as finite-difference schemes (Brennan and Schwartz (1976), Boyle and Schwarz (1977)) or Monte Carlo simulation as in Delbaen (1986), instead. All the works mentioned so far incorporate mortality risk by applying the law of large numbers. The survival probabilities are either calculated using mortality tables or by relying on a particular analytical distribution function. Still in the spirit of this traditional approach are the works by Aase and Persson (1994) and Bacinello and Ortu (1993a,b). Bacinello and Ortu (1993a) argue that due to mortality risk the fair and unique premium should be modelled endogenously by solving the resulting fixed-point problem. In contrast, Aase and Persson (1994) assume that survival probabilities are continuous which implies that benefits are due immediately upon the death of the insured. Additionally, they fix the units of the reference portfolio provided to the insured, which allows a closed-form solution even in the case of periodical premiums.

The modelling framework applied for pricing equity derivatives is much more complex than the models considered usually in the insurance literature. To capture empirically observed stylized facts in equity markets like volatility smiles, additional stochastic risk factors have to be included, e.g. stochastic volatility models like the Heston (1993) are suggested. When it comes to long maturity products an appropriate modelling of the financial market dynamics is essential, see Bakshi et al. (2000). Concerning, the choice of the stochastic models recent works on SLIPs use a Lévy pricing framework to capture the significant heavy-tails observed for the historical distribution of asset returns. Jaimungal (2004) uses the geometric Variance Gamma (VG) model to calculate the prices of the embedded option for a roll up and a cliquet style guarantee where the equivalent martingale measure is fixed. The author provides a detailed analysis of the differences between dynamic hedging parameters for the VG model and for the Black-Scholes model. He concludes that the differences can be dramatic and more sophisticated models like the VG model should be used for risk management. Kassberger et al. (2008), in particular, highlight the potential risk due to a misspecification of the stochastic process underlying the reference portfolio by considering different Lévy models. In contrast, Benhamou and Gauthier (2009) assume a Heston type model for the equity risk to model guaranteed minimum accumulation benefits. In addition, they account for stochastic interest rates by using the Heath et al. (1992) (HJM) affine interest rate model. They show that the impacts of stochastic interest rates and stochastic volatility are more pronounced on the embedded option's vega than on its delta.[7]

2.2.1.2 Extension to stochastic interest rates

The extension to stochastic interest rates is first studied in Bacinello and Ortu (1994). Most of the previous literature considered only deterministic interest rates. But due to the very long maturities of SLIPs stochastic interest rates can have a dramatic impact on pricing and risk management. Bacinello and Ortu (1994) consider a single premium contract and compare its value when the spot rates are modelled either by a Vasicek (1977) model or by a Cox et al. (1985) model. Nielsen and Sandmann (1995, 1996) extend the analysis to periodic premiums in a two-factor economy. In particular, they compare approximation results of the price of the Asian style option based on Vorst (1992) with Monte Carlo prices. Bacinello and Ortu (1996) consider contracts where the underlying reference portfolio consists of fixed income securities. An extension of the stochstic interest rate model to the general Heath-Jarrow-Morton model is provided in Bacinello and Persson (2000). Pelsser and Schrager (2004) link their lognormal model of the economy to a LIBOR market model by fitting to observed cap and swaption prices to guarantee a market consistent valuation of the insurance put. However, mortality fees and guarantee fees are determined exogenously and deducted from the account value similar to the price setting of Variable Annuities. Moreover, Pelsser and Schrager (2004) as well as Nielsen and Sandmann (2002a) provide economically meaningful and tight price bounds for the Asian style option relying on the conditioning approach dating back to Curran (1994) and Rogers and Shi (1995). Still, the choice of an appropriate stochastic interest rate model for long-term maturities is unsolved. Most of the existing literature considers a one-factor interest rate model

[7] The delta is the sensitivity of the option price w.r.t. the underlying, the vega w.r.t. the volatility.

which does not capture long run trends. A notable exception is Cairns (2004) developing an interest rate model which is supposed to fulfill the requirements of long-term contracts. However, an application to this type of life and pension insurance contracts is still missing.

The models and their underlying assumptions for the literature presented in this section are summarized in Table 2.2.

2.2.1.3 Extension to stochastic mortality

Another recent development is considering mortality risk to be stochastic. Evidence from the last decade shows that mortality probabilities change significantly over time. The so-called "rectangularization" is a stylized fact observed in the analysis of mortality data. Traditionally, the insurance company is supposed to be "risk neutral with respect to the mortality risk". This implies that no risk premium is charged. However, systematic changes in mortality rates have to be considered as a non-pooling risk. Thus, the law of large numbers does not apply anymore. Especially, the problem of longevity risk for pension insurance has been analyzed in increasing intense in the literature. However, it is beyond the scope of this work to give a detailed overview of stochastic mortality modelling. Instead, we want to point out that the number of contributions for structured life insurance products is very limited until now. Melnikov and Romaniuk (2006) study the effect of three different approaches for modelling mortality where one of the models is the recently developed method of Lee and Carter (1992) for fitting mortality and forecasting it as a stochastic process. They compare the three models in terms of the risk management using data from three countries. Finally, they ask the question whether insurance providers are aware that the risk management effectiveness potentially varies with the different mortality models. Another approach can be found in Nielsen et al. (2009) who shift the present age of the investor and take this as a conservative estimate for the pricing.

Author	Insurance Risk	Financial Risk		Embedded Option	Guarantee	Premium Payments
		Equity Risk	Interest Rate Risk			
Brennan and Schwartz (1976)	Law of Large Numbers	BS-Model/PDE Finite Differences	Deterministic Interest Rates	Insurance Call/Put	Maturity	Up-front and Periodical
Boyle and Schwartz (1977)	Law of Large Numbers	BS-Model/PDE Finite Differences	Deterministic Interest Rates	Insurance Put	Maturity	Up-front and Continuous Periodical
Delbaen (1986)	Law of Large Numbers	BS-Model/Martingale Approach Monte Carlo Simulation	Deterministic Interest Rates	Insurance Call/Put	Maturity	Up-front and Periodical
Bacinello and Ortu (1993a/b)	Law of Large Numbers	BS-Model	Deterministic Interest Rates	Insurance Call/Put	Endogenous Maturity Guarantee	Up-front and Periodical $\Rightarrow$ solution fixed point problem
Aase and Persson (1994)	Law of Large Numbers Continuous Death Probabilities	BS-Model PDE/Thieles Equation	Deterministic Interest Rates	Insurance Call/Put	No vs Maturity	Up-front and Periodical
Bacinello and Ortu (1994)	Law of Large Numbers	BS-Model	Vasicek Model for spot rate CIR Model	Insurance Call/Put	Maturity	Up-front Permium
Nielsen and Sandmann (1995)(1996)	Law of Large Numbers	BS-Model Approximation Methods	Vasicek Model	Asian Option	Maturity	Periodical
Bacinello and Ortu (1996)	Law of Large Numbers		Fixed-Income Securities/ CIR for spot rate	Insurance Call/Put	Maturity	Up-front
Bacinello and Persson (2000)	Law of Large Numbers	BS-Model	HJM Model	Insurance Call/Put	Maturity	Up-front time-dependent Periodical
Nielsen and Sandmann (2002)	Law of Large Numbers	BS-Model Conditioning Approach	Vasicek Model	Asian Option	Maturity	Periodical
Pelsser and Schrager (2004)	Exogenous Deduction	BS-Model Convexity Correction	LIBOR Market Model	Asian Option	Maturity	Periodical
Jaimungal (2004)	Law of Large Numbers	Variance Gamma Model	Deterministic Interest Rates	Insurance Put/Call	Maturity Cliquet	Periodical
Melinkov and Romaniuk (2006)	Gomberz, Makeham, Lee Carter	BS-Model	Deterministic Interest Rates	Insurance Option	Maturity	Up-front
Kassberger et al (2008)	Not considered	Lévy Model Esscher Transform	Deterministic Interest Rates	Forward starting call options	Cliquet	Up-front
Benhamou and Gauthier (2009)	Law of Large Numbers	Heston Model	HJM Model	Insurance Call/Put	Maturity Death	Up-front
Nielsen et al. (2009)	Transformed analytical mortality rates	BS-Model	Vasicek Model	Asian Option	Maturity Guarantee Schemes	Periodical

Table 2.2 Model choice.
The Table gives a chronological order of the most important research papers classified by the modelling framework and the characteristics of the considered contract.

2.2.2 Risk management

In general, all the literature considered so far is highly relevant as the valuation methods are based on hedging strategies. In most cases the hedging strategy is derived for an *ideal*, i.e. frictionless market.[8] However, this framework leads to veritable problematic aspects in practice as the idealistic assumptions are often violated and the derived hedging strategies are often not feasible or tractable. Indeed, as pointed out in Boyle and Hardy (1997), long-term embedded options are not traded in financial markets. Additionally, the hedging strategies derived in a specific model framework cannot be perfectly implemented if either practical trading restrictions or other regulations for the insurer are present. Moreover, the insurer faces model risk and the risk of a change in the death distribution which cannot be perfectly diversified. This implies that strategies which are based on one particular model, fail to be optimal if the true asset price dynamics/death probabilities deviate from the assumed ones. For this reason the insurer has to find a portfolio strategy which is meeting its liabilities i.e. minimizing the shortfall probability but which must also be feasible. Table 2.3 summarizes the risk inherent in these contracts, the hedging strategies and the resulting problems in practice. Already Brennan and Schwartz (1979) consider the problem which occurs if transaction costs are included precluding to implement the continuous riskless hedging strategy. They compare the discretized continuous investment strategy with no hedging, i.e. the investment share is completely credited to the underlying reference portfolio. Their results show that shortfall probability and in particular expected shortfall are significantly reduced due to the discrete hedging strategy compared to the naive strategy. In addition, they illustrate that the mean-variance optimal hedging strategy is a mix of the naive strategy and risk-reducing strategies with different discretizations. Boyle and Hardy (1997) tackle the question of how to build on reserves for SLIPs. They compare a stochastic simulation approach applied to the mutual fund development with option-based risk management strategies by taking into account items such

[8] Notice that this is not true for contract valuation which is based on Monte Carlo simulation, e.g. Asian options.

Risk	Hedging by	non-hedgable	Problems/Implemetation
Insurance Risk	perfect diversification	change in mortality distribution (longevity, mortality risk), misspecification of mortality distribution	no perfect diversification, non-pooling risk, early redemption
Financial market risk	Forward contracts/ short term options hedging strategy	non tradable embedded options model risk	investments if premiums are paid, discrete trading, prefinancing of periodic premiums

Table 2.3 Problems in risk management.

The Table lists the problems in risk management for insurance and financial risks.

as expenses, transaction costs and the impact of lapses. They find that for the long-term products the stochastic simulation approach has its merits compared to the option based approach. However, they conclude that this result does depend to a great extend on the particular contract investigated.

Another important issue concerns replacing the uncertainty in life expectancy with the expected one due to the law of large numbers. This implies that there is no mortality uncertainty within a portfolio of contracts. The options are bought with the correct maturities, i.e. the weighting of the different payoffs is based on the expected death probabilities. Møller (1998,2001) investigates how the combined financial and insurance risk can be hedged. The first paper only considers an up-front premium whereas the second paper extends to intermediate premiums and benefit payments. In both studies, he applies the concept of risk-minimizing hedging strategies based on Föllmer and Sondermann (1986) to unit-linked life insurance contracts. Thus, he conducts his analysis in an incomplete market setting and derives meaningful risk-minimizing strategies reacting to the financial and mortality risk. However, the expectation of the expected squared cost of the duplicating strategy is taken with respect to an adjusted probability measure and not to the real world measure which is the adequate choice for assessing the effectiveness of the risk management strategy. A different approach is provided in Møller (2003a,b) by using indifference pricing techniques, involving actuarial principals like the financial variance and standard deviation principle. Here, the latter paper applies the results of the first to different life insurance products, e.g. unit-linked contracts. In contrast, Jacques (2003) calculates the Value at Risk (VaR) for an individual equity-linked contract where the insurance company conducts five potential hedging strategies which do not necessarily coincide with the benchmark strategy. He concludes that if the uncertainty concerning the death of the insured remains a risk potential none of the analyzed hedging strategies dominates/outperforms the others.

Moreover, as mentioned above, model risk has to be of concern for an effective risk management. The influence of volatility misspecifications on arbitrage free option prices is discussed detailed in the literature, see Avellaneda et al. (1995) and El Karoui et al. (1998). An application of these results to insurance contracts and a following discussion of the hedging problems can be found in Mahayni and Schlögl (2008). In addition, they establish a conservative contract parameter setup and derive an effective risk management strategy.

Riesner (2006a), Riesner (2006b) as well as Vandaele and Vanmaele (2008) consider pricing and risk-minimizing hedging strategies for a unit-linked contract in a Lévy financial market model. Riesner (2006b) extends the results of Møller (1998, 2001, 2003a, 2003b) to general Lévy processes. In addition, he delivers an interpretation of the hedging risk under an investor's subjective probability measure and not only under some risk-neutral martingale measure. However, Vandaele and Vanmaele (2008) claim that the locally risk-minimizing strategy of Riesner is not correct and provide an alternative formula.

Even though, the following two papers consider guaranteed minimum death benefits with ratchet guarantees, most of the statements hold true for accumulation benefits. Coleman et al. (2006) calculate risk-minimizing hedging strategies using the underlying asset and standard options while allowing for jumps in the asset price dynamics and interest rate risk. Thus, the financial market is incomplete. Comparing the results they show that the effectiveness of the risk-minimizing strategy (underlying, option) is model dependent. Coleman et al. (2007) address the problem of transaction costs, limitations in the rebalancing frequency of the hedging portfolio and restrictions in liquidity due to the choice of the hedge instruments for the hedging of variable annuities. Their analysis is placed in an economy with stochastic volatility and jumps and deterministic interest rates. Table 2.4 summarizes the relevant literature on risk management and their contributions.

2.2.3 Additional riders and their pricing and risk management

A further point of interest concerns additional riders included in the design of the contracts. These encompass early redemption rights, i.e. the option to surrender or different guarantee styles and switch/shift rights. In Ekern and Persson (1996) several new types of unit-linked contracts are suggested, "with substantial potential for real life application". Amongst other additional riders they also address an option to switch portfolio weights between mutual funds the insured is invested in. Nowadays, such riders are included in Variable Annuities by Axa, Allianz or Swiss Life. The insured can switch at least four times a year the entire account value or the on-going premiums in other funds. This gives rise to an interesting optimal stopping problem which is first analyzed in Mahayni and Schoenmakers (2010) for a one-time switching right. In the center of their argumentation lies the reasoning that the insurance company has to take into account the highest possible guarantee value, i.e. investing in the worst case strategy which maximizes the embedded put-option. They show that the Black-Scholes model leads to a deterministic stopping time. Any, even a slight, deviation from the Black-Scholes assumptions leads no more to a deterministic stopping time. Here, the deterministic stopping time gives a lower bound. However, they argue that it is realistic to assume that an investor does not follow the optimal strategy for mainly two reasons. On the one hand, a risk averse investor prefers a diversifying strategy instead of the worst case (non-diversifying) strategy. On the other hand, the investor might not be able to implement the optimal strategy due to model risk or lack of knowledge. Thus, any other strategy followed by the policyholder will result in lower costs for the insurance company and therefore in sunk costs for the insured. An extension to multiple

Author	Insurance Risk	Financial Risk	Risk Management Approach	Guarantee	Premium Payments
Brennan and Schwartz (1979)	Law of Large Numbers	BS-Model Transaction Costs/ Discrete Rebalancing	Investment in Fund vs Discrete BS-Hedge / Different Rebalancing Frequencies	Maturity	Periodical
Boyle and Hardy (1997)	Mortality of 0.5% p.a. Lapses of 5% p.a.	Simulation simulation approach BS- Model	Reserving by Simulation by Discrete BS-Hedge	Maturity	Up-front Periodical
Møller (1998)	Combined Insurance and Financial Risk Poisson Distribution/ BS-Model		Risk-Minimizing Hedging	Maturity	Up-front
Møller (2001)	Combined Insurance and Financial Risk Poisson Distribution/ BS-Model		Risk-Minimizing Hedging	Maturity	Periodical
Møller (2003a) Møller (2003b)	Combined Insurance and Financial Risk Poisson Distribution/ BS-Model		Indifference Pricing	Maturity	Periodical
Jacques (2003)	Individual Death Probabilities	BS-Model	Different Investment Strategies (VAR)	Maturity	Up-front
Riesner (2006)	Combined Insurance and Financial Risk Poisson Distribution/ Lévy Model		Risk-Minimizing Strategies	Maturity	Periodical
Mahayni and Schlögl (2008)	Not Considered	BS-Model Vasicek Model	Conservative Pricing Market Consistent Risk Management	Cliquet	Periodical
Vandaele and Vanmaele (2008)	Combined Insurance and Financial Risk Poisson Distribution/ Lévy Model		Risk-Minimizing Strategies	Maturity	Periodical

Table 2.4 Risk management.
The Table summarizes the current literature on risk management of contracts with minimum return guarantees.

stopping rights or practical relevant stochastic volatility models is still missing.

Another feature often included in structured life insurance products is the option to surrender. The option to surrender can be understood as an early-exercise feature. The insured is allowed to terminate the contract before maturity and receives a so-called surrender value. The surrender value is the accumulated fund value until surrender less a certain surrender fee. Generally, this option is only provided if the insured receives a benefit both in the case of death and the case of survival. In a contingent claim framework the pricing of this right is similar to the pricing of American style options or in case of discrete surrender rights of Bermudan options. The first study considering the pricing of an American style surrender guarantee (continuous time trading) is conducted by Grosen and Jørgensen (1997) who abstract from any mortality risk in case of an up-front premium. Besides the valuation of the American guarantee, the main focus is on the redistribution of wealth for insureds investing in the same fund but with different guarantees. However, in the absence of *correct valuation* of the guarantees, the funds can hardly be distributed fairly among the different policyholder. Therefore, they also consider different exit fees (penalties) which compensate the insurance company if the insured terminates the contract before maturity and do not treat surrender in the same way as death. The bulk of the following literature either focuses on better numerical approximation techniques of the Bermudan like option and/or also include mortality risk. Bacinello (2005) argues, for instance, that the introduction of mortality risk increases the complexity of the problem to a great extend as there is a continuous interaction between mortality and financial risk factors. To surrender the contract involves continuous/discrete comparisons between the surrender value and the fund value which also depends on the death of the insured. Moreover, in the case of periodic premiums the on-going premiums depend on the death and the surrender of the contract. Bacinello (2005) prices the contract with surrender option in a Cox et al. (1985) setup and analyzes necessary and sufficient conditions for existence and uniqueness of the fair premium. In contrast, Shen and Xu (2005) adopt a PDE approach where the surrender option problem is modelled in terms of a free-boundary problem which implies numerical procedures often too complex without simplifying assumptions. A different methodology is employed by Bacinello et al. (2008) and Bacinello et al. (2009) who rely on Least-Squares Monte Carlo Simulation. Both papers deal with stochastic volatility, jumps in asset prices as well as randomness in the force of mortality. The second paper extends the first by refining the valuation procedures and testing for two algorithms dependent on the generality of the setup (assumption on insured's time of death). A similar but in two aspects differing approach is provided by Bernard and Lemieux (2008). They rely on the usual assumption that the mortality risk is independent of the financial risk, thus mortality rates do not have to be simulated. Additionally, they use control variate techniques and perform simulations using quasi-random sampling which should make the simulation more efficient and accurate. We think that it is worth mentioning that the mentioned papers consider the insured to be rational, i.e. the contract is surrendered if certain economic events occur. In contrast, classical actuarial science estimates surrender exogenously from historical data on the lapse rate, see e.g. Anzilli et al. (2004). However, historical lapse rates can significantly underestimate the true surrender of the individuals. For this reason, the focus for the design of such contracts should be based on strategies which cover the hedging costs of the insurance company but reduce the loss in risk capital of the insured.

For ratchet or reset features and/or combinations of the two we have to deal with complex exotic options. As mentioned above the interpretation of the ratchet style differs among the different contract designs. In particular, we have to differentiate between the literature who translate ratchet as "cliquet" and the one where only the guarantee value is ratcheted up. The first interpretation is especially typical for equity-indexed annuities and segregated funds. One of the first who considers the ratchet option in the context of insurance contracts is Tiong (2000). He uses Esscher transforms in a complete market to value various embedded options in EIA including ratchet options and their extension.[9] The purpose of this paper is the pricing of these products and the comparison of the results to gain more insight into the ratchet mechanism. In

[9] Esscher transforms are often used in actuary mathematics. In this context the Esscher transforms are used to determine the equivalent martingale measure.

contrast, Hardy (2004) considers embedded ratchet features distinguishing between a compound annual ratchet contract and a simple annual ratchet contract relying on the unique equivalent martingale measure in a complete market. The former can be priced in closed form whereas the latter has to be approximated numerically. Hardy (2004) implements a non-recombining lattice to value the simple annual ratchet feature. In addition, she includes a ceiling rate and a floor rate at maturity. The floor rate can be interpreted as an additional roll-up guarantee. However, no closed-form solution does exist to price the combined contract. Therefore, she compares the non-recombining lattice with Monte Carlo simulation. Kijima and Wong (2007) extend the analysis by incorporating stochastic interest rates via an extended Vasicek model whereas Jaimungal (2004) values the compound ratchet style guarantee in a Variance Gamma model. The main focus of both papers is on the pricing of these contracts. Mahayni and Sandmann (2008) argue that if the excess return is determined annually then it must be accumulated until the maturity of the contract. They distinguish between a stochastic and a deterministic accumulation factor and show that the well-known robustness result of the Black-Scholes model is not valid in case of the deterministic accumulation factor which has severe impact on the hedging effectiveness.

The other version of a ratchet style guarantee is that the included option resembles a reset option, e.g. Cheng and Zhang (2000).[10] At stipulated dates the guarantee is raised to the greater of the account value and the initial guarantee which is usually equal to the investment in the fund. Thus, the resulting option is affected by the entire price process of the underlying fund. At maturity the insured receives the maximum of the fund value at maturity and the highest value of the fund at the reset dates. In addition, this level is also guaranteed even if the underlying portfolio falls back below these levels before the option expires. A similar contract design can be found in Hipp (1996). He additionally includes a roll-up guarantee at maturity. However, a valuation and detailed analysis of this kind of guarantee is not provided up to now.[11]

[10] This is typical for Variable Annuities.

[11] A general pricing framework which also includes ratchet guarantees can be found in Bauer et al. (2008).

Author	Insurance Risk	Financial Risk	Embedded Option	Guarantee	Premium Payments
Option to Switch					
Mahayni and Schoenmakers (2010)	Not considered	BS-Model - Lévy-Model - Regime Switching	American Compound Put Option	Maturity	Up-front
Option to Surrender					
Grosen and Jørgensen (1997)	Not considered	BS-Model	American Put Option	Maturity	Up-front
Bacinello (2005)	Individual death - no risk premium charged	Binomial Lattice	American Put Option	Maturity	Up-front Periodical
Shen and Xu (2006)		BS-Model - PDE Approach	American Put Option	Maturity	Up-front - Periodical
Bernard and Lemieux (2008)	Law of Large Numbers	BS-Model - LSMC	American Put Option	Cliquet	Up-front - Periodical
Bacinello et al. (2008)		BS-Model - LSMC	American Put Option	Maturity	Up-front and Periodical
Bacinello et al. (2009)		BS-Model - LSMC	American Put Option	Maturity	Up-front and Periodical
Guarantee Styles					
Tiong (2000)	Not considered	BS-Model - Esscher Transform	Forward Starting options	Cliquet -Cliquet + Maturity	Up-front
Hardy (2004)	Not considered	BS- Model - Monte Carlo	Forward Starting options	Cliquet - Cliquet + Maturity	Up-front
Kijima and Wong (2007)	Not considered	BS-Model - Extended Vasicek	Forward Starting options	Cliquet -Cliquet + Maturity	Up-front
Jaimungal (2004)	Not considered	Lévy Model Variance Gamma Model	Insurance Put/Call	Maturity Cliquet	Periodical
Mahayni and Sandmann (2008)	Not considered	BS-Model - Stochastic Interest Rates Convexity Adjustment	Forward Starting options	Cliquet	Periodical
Hipp (1996)	Not considered	BS-Model	Ladder Option	Ratchet	Up-front
Windcliff et al. (2001b)		BS-Model - Monte Carlo	Shout options	Reset	Periodical
Windcliff et al. (2001a)	Mortality Data	BS-Model - Backward PDE	Shout options - Hedging Heuristics	Reset	Periodical
Boyle et al. (2001)	Not considered	BS-Model	Shout options/Heuristics	Reset	Periodical
Dai et al. (2004)	Not considered	BS-Model	Shout options/Heuristics	Reset	Periodical
Dai and Kwok (2005)	Not considered	BS-Model - Free Boundary Problem	Shout options/Heuristics	Reset	Periodical

Table 2.5 Additional riders.
The Table gives an overview on the literature concerning the valuation and risk management of additional riders.

Coming to the reset style guarantee we are again, at least theoretically, in an optimal stopping problem. The holder of this additional rider is allowed to reset the guarantee level up to multiple times during the life of the contract. In addition, some contracts not only reset the strike of the option but also extend the maturity of the contract. For instance, recently traded Variable Annuities include guaranteed minimum accumulation benefits running ten years where the additional rider is provided that the insured can "shout" such that the guarantee is changed and the maturity is renewed for another ten years. Thus, the embedded options are so-called shout options with an additional maturity extension.[12] We have to differentiate between pure finance literature addressing reset options in general and the insurance literature which also accounts for insurance risk. Concerning the latter stream of the literature we refer to Windcliff et al. (2001b) who evaluate the embedded option relying on the numerical solution of a set of linear complementary problems. Moreover, they also compare guarantee prices of traded contracts with their theoretical results and identify a significant underpricing. For their numerical examples the impact of mortality on prices is almost insignificant. Windcliff et al. (2001a) extend the investigation of price reduction due to heuristically chosen reset dates and contract designs which are more effective for risk management purposes or might explain the observed *too low* price setting of the insurance companies. Without postulating completeness the papers by Boyle et al. (2001), Dai et al. (2004) and Dai and Kwok (2005) consider the pricing of shout options but without taking into account mortality risk but focusing on different numerical procedures.

[12] Here, applies the same argumentation as above that the policyholder usually does not follow the overall optimal strategy. However, from a risk management perspective it is necessary to assess the highest possible guarantee value to stay on the safe side. Nevertheless, when it comes to the customer's perspective the effect of the costs which he pays for a feature which he cannot use or does not want to use, could rise the question of how the insurer has to modify the design of the contract in order to reduce hedging expenses and meet customer needs.

2.3 Interaction between insurance company and insured

As a consequence of the flexibility in the design of SLIPs, the interaction between insurance company and insured attain an increasing relevance compared to the classical life and pension insurance products. On the one hand, the insured's choice of certain features impacts the risk management decisions of the insurance provider. On the other hand, a well performing company with well-designed products attracts insurance takers and influences their decision in favor of a company/product. Caused by the additional riders in SLIPs, the insurance companies' awareness of the insurance takers' preferences and behavior gains importance. In particular, the analysis of the implications for the risk management and future contract design become relevant to ensure the long-term competitive position of the insurance provider. This strain of thoughts is also reflected in the current research literature.

The first question which presents itself is whether the insured actually wants products with minimum interest rate guarantee or whether guarantees are only present because of regulatory requirements and marketing effects. The latter can be regarded as a purely behavioral question as it concerns the power of directing the decisions of the people by presenting facts influencing the insureds' decision taking. An investigation of this issue would require economic laboratory experiments and interviewing of costumers. However, in a first step it is necessary to answer the question as to whether one can explain guarantees by considering the optimal design of an insurance contract from the perspective of the insured.

Boyle and Tian (2008) solve the optimization problem of an insured who maximizes expected utility of terminal wealth under the constraints that the terminal wealth must remain above an exogenous guarantee with some probability. They argue that such a design better fits investor's needs than existing products like equity-indexed annuities. A different approach is taken in Døskeland and Nordahl (2008) considering participating life insurance contracts. Nevertheless, the main points are also valid in the concept of a SLIP. Comparing different contracts, they can explain guarantees by applying cumulative prospect theory but not by CRRA utility. Branger et al. (2010) in turn argue that under the CRRA utility a guarantee can never be optimal but they derive the investment strategy and contract design which leads to the smallest utility loss by including an exogenous guarantee in the optimization problem. Branger et al. (2010) show that the optimal design is given by a Variable Annuity where the deduction from the account value is chosen to match the price of the embedded put option. The optimal underlying strategy is given by a constant mix. Although, this design leads to very small utility losses, the hedging problem remains unsolved. In particular, due to additional riders the complexity of hedging and the impact on the design increases once more. A first study which considers the impact on the investors utility by the option to switch investment decisions dynamically is given in Mahayni and Schneider (2010). They show that there exists an incentive to deviate from the optimal diversified strategy to a riskier strategy due to the worst case pricing of the insurance company. Thus, the question arises of how a contract design does look like which is easy to hedge and fits more or less the investors' preferences. We think that it is necessary to think of a simultaneous approach to design such contracts and to ensure that their risk management is tractable.

A study which sheds light on the question of how valuable guarantees are for insureds is presented by Gatzert et al. (2011 (forthcoming). Subjective prices are obtained via an online questionnaire and compared to fair prices which turn out to be significantly higher. Even though, this gives insight into insured's perception of guarantees, broader studies are missing. In addition, an empirical investigation of the insured's investment decisions would be of interest. A comparison between the optimal strategies and the strategies conducted by the insurance company is still owing, too.

2.4 Conclusion

This chapter gives an overview on the existing literature of structured life insurance contracts and points out interesting research questions which still have to be tackled. In particular, the question whether the design of currently offered life insurance contracts actually suit the retail investors preference is put forward. The remaining two chapters try to shed light on this question.

Chapter 3

Optimal Design of Insurance Contracts with Guarantees [1]

3.1 Introduction

In the last chapter we subsumed guaranteed minimum accumulation benefits as well as the different kinds of equity–linked products under the name *structured life insurance products*. The basic difference lies in the deduction of the guarantee. In fact, this gives rise to different guarantee schemes. The participation surplus scheme and the contribution guarantee scheme. [2]

As a consequence, the insured thus has two choices to make: he has to decide on the exact form of the guarantee, i.e. the design of the insurance product, and he has to decide on the investment strategy underlying the insurance contract. The main focus of this chapter concerns the question which contract design is optimal for an insured who saves towards retirement and maximizes expected utility.

The set of admissible contracts is restricted by the condition that the contract is fair. In general, there are no restrictions on the investment strategy. Any shortfall of the portfolio underlying the insurance contract with respect to the guaranteed level is covered by a put option on this portfolio. We assume that the market is complete, i.e. there exists a self-financing and duplicating strategy for the put options under consideration. The initial investment into this strategy is the price of the option, and the replicating

[1] This chapter is based on joined work with Nicole Branger and Antje Mahayni published in Insurance: Mathematics and Economics.

[2] The naming of the guarantee schemes is along the lines of Nielsen et al. (2009).

strategy allows for a perfect hedge of the option.[3] Given the prices of the put options, we can define the set of admissible contracts, i.e. the feasible combinations of guaranteed rate and participation rate. Stated differently, for an exogenously given guarantee, the choice of the investment strategy also implies the admissible participation rate.[4] Therefore, we first illustrate the fair contract specification for several combinations of investment strategies and guarantee schemes. The investment strategies include buy-and-hold strategies, constant mix strategies, and constant proportion portfolio (CPPI) strategies. We combine these strategies with the two guarantee schemes and show that the fair contract parameters crucially depend on both choices.

We assume that all fair contracts are feasible for the insured. In the next step, we take the perspective of the insured who has to decide on the optimal combination of investment strategy and guarantee scheme. For a CRRA insured, this optimal solution is given by a constant mix strategy combined with a contribution guarantee scheme. For a HARA insured with a subsistence level equal to the guarantee, the optimal choice is given by a CPPI strategy, without the need to include an additional guarantee scheme.

Finally, we study the utility losses of a CRRA insured due to the exogenous guarantee and due to restricting the investor to a given set of investment strategies and guarantee schemes. If this set includes the constant mix strategy and the contribution guarantee scheme, the utility losses of the insured are only due to the exogenous guarantee. Our main focus is then on the additional utility losses due to choosing suboptimal investment strategies, suboptimal guarantee schemes, or both. We illustrate our findings by some numerical examples.

There are various strands of related literature. Amongst them are papers focusing on one or more of the following topics: pricing of embedded options, portfolio planning,

[3] In an incomplete market, one has to make assumptions on the pricing of non-spanned risk factors, and a perfect hedge is in general not possible any more. Perfect hedging is also impeded by model risk and market frictions. For financial contracts, the robustness of hedging strategies under model risk is considered in Avellaneda et al. (1995) and El Karoui et al. (1998). An application to insurance contracts can, for example, be found in Mahayni and Schlögl (2008).

[4] We do not aim to explain the existence of guarantees, but rather take them as given. The analyzes of insurance contracts with guarantees can be motivated by regulatory requirements.

dynamic portfolio insurance strategies, participating contracts with different distribution mechanisms, variable annuities and equity–linked insurance products. Without postulating completeness we refer to the following works. For the literature on the pricing of embedded options in life insurance portfolio we refer to Chapter 2.

In the context of dynamic portfolio insurance strategies, we mainly refer to papers on CPPI. The properties of continuous–time CPPI strategies are studied extensively in the literature, cf. Bookstaber and Langsam (2000) or Black and Perold (1992). Draw–backs of the CPPI approach are the so–called cash–lock–cage and the gap risk. While the former refers to the possibility that in downward moving markets the exposure is, at an early stage, reduced to zero and stays there, the latter is even more serious because it refers to a violation of the guarantee. Basically, the CPPI method fails to meet the guarantee if the portfolio can not be rebalanced *fast enough*, i.e. due to price jumps or trading restrictions. An analysis of gap risk is, for example, provided in Cont and Tankov (2009) and Balder et al. (2009).[5] A comparison of option based and constant proportion portfolio insurance is given in Bertrand and Prigent (2002a) and Zagst and Kraus (2009).[6]

Concerning the literature on portfolio planning, references include Merton (1971) who solves the portfolio planning problem for a CRRA investor. Kim and Omberg (1996), Barberis (2000), and Wachter (2002) consider optimal portfolios when stock returns are predictable, while Liu and Pan (2003) and Liu et al. (2003) study models with stochastic volatility and jumps. Sørensen (1999), Brennan and Xia (2000, 2002), Munk and Sørensen (2004) deal with stochastic interest rates. Basak (2002) shows that adding a subsistence level to the problem leads to portfolio insurance strategies.

Until now, less work has been done w.r.t. the optimality of insurance contracts with interest rate guarantees. Related literature also includes Jensen and Sørensen (2002), Huang et al. (2008), Milevsky and Kyrychenko (2008), Boyle and Tian (2008) and

[5] Cont and Tankov (2009) consider gap risk caused by jumps in the stock price. Balder et al. (2009) analyze the impact of trading restrictions and also capture the effects of transaction costs.

[6] Further papers analyze the effects of jump processes, stochastic volatility models and extreme value approaches on CPPI strategies, cf. Bertrand and Prigent (2002b), Bertrand and Prigent (2003).

Døskeland and Nordahl (2008). The paper which is most similar to our work is Gatzert et al. (2009). They also take the perspective of the insured and apply risk neutral evaluation techniques to identify the set of admissible contracts. In contrast to our work they consider participating life insurance contracts and compare them assuming mean-variance preferences. In addition, they also take alternative investment opportunities besides the chosen insurance contracts into account.

The outline of the chapter is as follows. In Section 3.2, we define the two different investment guarantee schemes which can be combined with any investment strategy. Section 3.3 gives the model setup, introduces the relevant investment strategies and discusses the fair pricing of the structured life insurance contracts. In Section 3.4, we solve the optimization problem of the insured and determine the utility losses due to the guarantee and due to the restriction to specific strategies and guarantee schemes. Section 3.5 concludes.

3.2 Product design

The *structured* insurance contract is designed to guarantee the investor a certain amount of money even if the benchmark investment, i.e. mutual fund, falls below a floor. Besides the guarantee level, the investment strategy underlying the mutual fund plays a key role in the design of the contract. The insured can usually choose between a wide range of different investment strategies. Examples include pure bond strategies, pure stock strategies, constant mix strategies, investments into a basket of stocks or CPPI strategies.

The premiums paid by the insured are denoted by P. We assume that he pays the whole premium P at inception of the contract $t_0 = 0$. The payoff from the insurance contract depends on some underlying investment strategy. Let $I(t,P)$ denote the value at time t generated by the investment strategy with no in- or outflows and an initial investment equal to P. We assume that the retirement date of the insured is known and that there is no mortality risk, so that the maturity of the contract coincides with T.

These simplifying assumptions make it easier to focus on our main point, which is the optimality of different guarantee designs and different investment strategies.

The guaranteed amount $G(T,P,g)$ depends on the premium payment and a guaranteed interest rate g. It is defined as

$$G(T,P,g) := Pe^{g(T-t_0)}. \tag{3.1}$$

The payoff of the insurance contract is then a function of $G(T,P,g)$ and $I(T,P)$. We consider two guarantee schemes. The first is in the spirit of a guaranteed minimum accumulation benefit. The other one is conventionally used with equity–linked contracts. Here, we adopt the terminology of Nielsen et al. (2009) and refer to the schemes as the contribution guarantee scheme and the participation surplus scheme.[7]

Definition 3.1 (Guarantee Schemes). The payoff at time T depends on the value of the investment strategy at time T, the guarantee, the participation rate α ($\alpha \in [0,1]$), and the exact kind of guarantee scheme. The guarantee schemes and associated payoffs are

(1) contribution guarantee scheme with payoff $\mathscr{S}_{CG}(I(T,P),\alpha,g)$

$$\begin{aligned} \mathscr{S}_{CG}(I(T,P),\alpha,g) &:= \max\{\alpha I(T,P), G(T,P,g)\} \qquad (3.2)\\ &= G(T,P,g) + [\alpha I(T,P) - G(T,P,g)]^+ \\ &= \alpha I(T,P) + [G(T,P,g) - \alpha I(T,P)]^+, \end{aligned}$$

(2) participation surplus scheme with payoff $\mathscr{S}_{PS}(I(T,P),\alpha,g)$

$$\begin{aligned} \mathscr{S}_{PS}(I(T,P),\alpha,g) &:= G(T,P,g) + \alpha\,[I(T,P) - G(T,P,g)]^+ \qquad (3.3)\\ &= \alpha I(T,P) + (1-\alpha)G(T,P,g) + \alpha\,[G(T,P,g) - I(T,P)]^+. \end{aligned}$$

The contribution guarantee scheme and the participation surplus scheme promise the investor a payment of at least $G(T,P,g)$. Figure 3.1 shows the payoffs from the

[7] Nielsen et al. (2009) also consider a third guarantee scheme, the so–called investment guarantee scheme. However, we exclude this scheme because it is not consistent with an exogenously given guarantee.

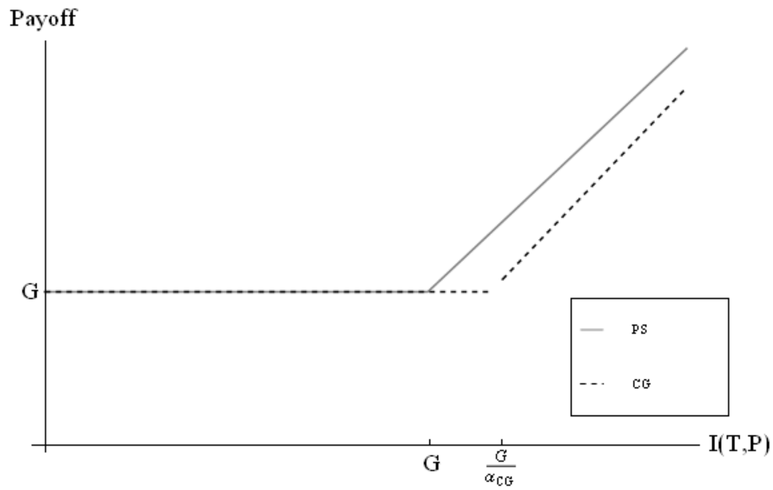


Fig. 3.1 Payoff of guarantee schemes
The figure shows the payoff of the contribution guarantee scheme (dashed line) and the participation surplus scheme (solid line) as a function of $I(T,P)$. The terminal guarantee is G.

insurance contract as a function of $I(T,P)$. If the value of $I(T,P)$ can fall below the guarantee level, the guarantee is valuable for the insured. In this case, the participation rate α has to be below one for the contract still to have a value equal to the initial premium P, i.e. to result in a fair contract specification. In a complete market the t_0–value of the T–payoffs resulting from the guarantee schemes can be represented by the expected discounted payoff under the uniquely defined equivalent martingale measure $\mathbb{P}^*$. Using the generalized principal of equivalence gives the following definition of a fair contract.

Definition 3.2 (Fair contract). A contract is called fair if the no-arbitrage price of the payoff $\mathscr{S}_w$ is equal to the contribution P of the insured. In particular, for $w \in \{CG, PS\}$ the tuple (α_w^*, g_w^*) satisfying

$$P = \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}\mathscr{S}_w(I(T,P),\alpha_w^*,g_w^*)\right] \tag{3.4}$$

is called fair.

We give the fair participation rate α of the guarantee schemes as a function of the guaranteed interest rate g.

Proposition 3.1 (Fair participation rate α_w^*). *For $w \in \{CG, PS\}$ it holds*

$$\alpha_{CG}^* \quad \text{is the solution of } P = \beta P + Put_{t_0}^{\beta I,G} \tag{3.5}$$

$$\alpha_{PS}^* = \frac{P - e^{-rT} G(T,P,g)}{P - e^{-rT} G(T,P,g) + Put_{t_0}^{I,G}} \tag{3.6}$$

where $Put_{t_0}^{\beta I,G} := \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}(G(T,P,g) - \beta I(T,P))^+\right]$ with $\beta \in \{\alpha, 1\}$ and where $\mathbb{P}^*$ denotes the equivalent martingale measure.

Proof. Follows immediately from Definition 3.1 and Definition 3.2. □

Note that both, the guarantee level and the terminal value of the underlying investment strategy $I(T,P)$ are proportional to P. The same holds true for the values of the embedded puts.[8] This implies that the fair participation rate α^* is independent of P. It depends on the guarantee level g, the maturity of the contract, and on the characteristics of the investment strategy which defines the terminal payoff $I(T,P)$. α^* also depends on the guarantee scheme w where $w \in \{CG, PS\}$.

For the sake of completeness we summarize in the following the model independent properties of fair participation rates and guarantee rates which are derived in Nielsen et al. (2009). For identical guaranteed interest rates, it holds that $\alpha_{CG} \geq \alpha_{PS}$. For identical participation shares, it holds that $g_{CG} \geq g_{PS}$. Furthermore, the participation share α is a decreasing function of the guaranteed interest rate g, independent of the model setup.

In the following, we will analyze which combination of guarantee scheme and underlying investment strategy is optimal for the insured. We do not give a justification for the guarantee, but assume that the lower bound G for the terminal payoff is given exogenously.[9]

[8] We assume a model for the stock price for which option prices are homogeneous of degree one in the stock price and the strike price, cf. Bergman et al. (1996).

[9] For the ease of notation, we omit the arguments T, P and g of G in the following.

3.3 Financial market and investment strategies

3.3.1 Complete financial market model

We consider a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{t\geq 0})$. The complete and arbitrage-free financial market consists of the stock S and the risk-free asset D. The price processes under the real world measure $\mathbb{P}$ are given by

$$dS_t = S_t\left(\mu_S dt + \sigma_S dW_t\right), \qquad S_0 = s, \tag{3.7}$$

$$dD(t,T) = D(t,T)rdt, \qquad D_0 = e^{-rT}, \tag{3.8}$$

where W denotes a standard Brownian motion. The risk-free rate r and the parameters μ_S and σ_S are constant. To determine the fair parameters of the insurance contract (α^*, g^*), it is necessary to price the embedded option. The price dynamics under the equivalent martingale measure $\mathbb{P}^*$ are given by

$$dS_t = S_t\left(rdt + \sigma_S dW_t^*\right) \tag{3.9}$$

$$dD(t,T) = D(t,T)rdt. \tag{3.10}$$

W^* is a one-dimensional Brownian motion under the equivalent martingale measure $\mathbb{P}^*$.

As stated above we want to analyze the design of a structured insurance product where the insured can choose between a range of investment strategies. We consider strategies investing in the equity S and the risk free asset D. A continuous–time investment strategy or saving plan for the interval $[0,T]$ can be represented by a predictable process $(\phi_t)_{0\leq t\leq T}$ where $\phi_t = (\phi_{t,S}, \phi_{t,D})$ denotes the number of shares of the asset S and the risk–free investment D. W.l.o.g., we restrict ourselves to strategies which are self–financing after the initial investment V_0, where a strategy ϕ with value process $V_t(\phi) = \phi_{t,S}S_t + \phi_{t,D}D(t,T)$ is called self–financing iff

$$V_t = V_0 + \int_0^t \phi_{u,S}\, dS_u + \int_0^t \phi_{u,D} dD(u,T). \tag{3.11}$$

Alternatively, the investment strategies can be represented by the investment fractions $\pi_t = (\pi_{t,S}, \pi_{t,D})$. Here, the self–financing requirement means that the investment fractions sum to one. In summary the dynamics of the portfolio value V are given by

$$\begin{aligned} dV_t &= \phi_{t,S}\, dS_t + \phi_{t,D}\, dD(t,T) \\ &= V_t\left(\left[\pi_t(\mu - r) + r\right] dt + \pi_t \sigma_S\, dW_t\right), \quad V_0 = x_0. \end{aligned} \tag{3.12}$$

where $\pi_t = \pi_{t,S}$ denotes the investment fraction of the stock.

3.3.2 *Investment strategies*

We consider several investment strategies or dynamic asset allocation strategies which are of theoretical and practical relevance. The standard strategies we look at are *Buy and Hold*, *Constant Mix*, and *CPPI*. In the above model setup all strategies are path–independent, i.e. their terminal payoff only depends on the stock price at maturity.

Strategies			
name	abbreviation	number of assets	investment fraction
buy and hold	*B&H*	$\phi_{t,S}^{B\&H} = \frac{\pi_0 V_0}{S_0}$	$\pi_t^{B\&H} = \pi_0 \frac{\frac{S_t}{S_0}}{\frac{V_t}{V_0}}$
constant mix	*CM*	$\phi_{t,S}^{CM} = \frac{\pi_0 V_t}{S_t}$	$\pi_t^{CM} = \pi_0$
constant proportion portfolio insurance	*CPPI*	$\phi_{t,S}^{CPPI} = \frac{m(V_t - e^{-r(T-t)}G)}{S_t}$	$\pi_t^{CPPI} = \frac{m(V_t - e^{-r(T-t)}G)}{V_t}$

Table 3.1 Investment strategies.
The table gives the initial number of assets and the initial investment fraction for the three strategies buy-and-hold, constant mix and CPPI. The buy-and-hold strategy is usually defined by the number of stocks, the constant-mix strategy is defined by the investment fraction, and the CPPI strategy is specified by the multiplier *m*.

The investment fraction and the number of assets which are implied by each of these strategies are summarized in Table 3.1. A constant investment fraction as prescribed in a constant-mix strategy implies that the number of assets which are held is state–

dependent. In contrast, a static buy-and-hold strategy is associated with a constant number of assets, but a state dependent investment fraction, c.f. Table 1. In practice, the CM strategy corresponds to an insured who can choose between mutual funds with different asset allocation rules and thus with a different riskiness. The CPPI strategy is supposed to give a terminal payoff which will never fall below the guarantee level G. It thus belongs to the class of dynamic portfolio insurance strategies.[10] For a given guarantee G the only strategy parameter is the so–called multiplier m. The basic idea is to invest m times the cushion $C_t = V_t^{CPPI} - Ge^{-r(T-t)}$ in the risky asset. Normally a CPPI implies that $m \geq 2$. However, it already results in a convex payoff for $m > 1$.[11] In the special case that $G = 0$, the CPPI is also a CM strategy. For $m = 1$, the CPPI is equal to a Buy and Hold Strategy.

Lemma 3.1 (Portfolio value). *In the model defined by Equations (3.7) and (3.8), the portfolio value V_t^b at time t with $b \in \{B\&H, CM, CPPI\}$ is given by*

$$V_t^b = \begin{cases} V_0^b \left(\pi_0 \frac{S_t}{S_0} + (1-\pi_0)e^{rt} \right) & b = B\&H, \\ V_0^b h_t(\pi_0) \left(\frac{S_t}{S_0} \right)^{\pi_0} & b = CM, \\ Ge^{-r(T-t)} + \left(V_0^b - Ge^{-rT} \right) h_t(m) \left(\frac{S_t}{S_0} \right)^m & b = CPPI, \end{cases} \tag{3.13}$$

where $h_t(\pi_0) = e^{(1-\pi_0)(r+\pi_0 \frac{1}{2}\sigma^2)t}$.

The insured can choose one of these three investment strategies and decide on the initial investment fraction of the stock.[12] His initial investment is given by P. The value of the investment strategy at time T is then equal to

$$I(T,P) = P \frac{V_T^b}{V_0^b} \text{ where } b \in \{B\&H, CM, CPPI\}. \tag{3.14}$$

[10] Besides the CPPI, a prominent example of a portfolio insurance is the option based portfolio insurance (OBPI). In a complete market the OBPI is based on a self-financing and duplicating strategy in S and D replicating the embedded option. The perfect hedge is impeded by market frictions and model risk. For a detailed comparison of OBPI and CPPI see, e.g. Bertrand and Prigent (2002a).

[11] For the desirable property of convex payoff functions of portfolio insurance strategies see for example Perold and Sharpe (1988).

[12] In case of a CPPI, the multiplier is then equal to $m = \frac{\pi_0 V_0}{C_0}$.

This value is the underlying for the guarantee schemes summarized in Definition 3.1.

3.3.3 Pricing of the embedded options

We now turn to the prices of the options embedded in the guarantee schemes of Definition 3.1. With Equation (3.14) the embedded options can also be interpreted as options on the payoff V_T^b.

Proposition 3.2 (Fair pricing of the embedded put option). *The price $Put_{t_0}^{V^b,G}$ of a put with terminal payoff $(G(T,P,g)-V_T^b)^+$ is given by*

1. *For $V^{B\&H}$*

$$Put_{t_0}^{V^{B\&H},G} = 0 \cdot 1_{\left\{\phi_{0,D}^{B\&H} \geq G\right\}} + \phi_{0,S}^{B\&H} PO\left(t_0, S_0, 1, K^{B\&H}, T\right) 1_{\left\{\phi_{0,D}^{B\&H} < G\right\}} \tag{3.15}$$

$$\text{where } K^{B\&H} = \frac{G-\phi_{0,D}^{B\&H}}{\phi_{0,S}^{B\&H}}.$$

1 denotes the indicator function. $\phi_{0,D}^{B\&H}$ and $\phi_{0,S}^{B\&H}$ are as in Table 3.1.

2. *For V^{CM}*

$$Put_{t_0}^{V^{CM},G} = \vartheta PO(t_0, S_0, \pi_0, K^{CM}). \tag{3.16}$$

where $\vartheta = \frac{V_0^{CM} h_T(\pi_0)}{S_0^{\pi_0}}$ and $K^{CM} = \frac{G}{\vartheta}$.

3. *For V^{CPPI} and with the additional restriction $V_0^{CPPI} = P$, it holds that*

$$Put_{t_0}^{V^{CPPI},G} = 0. \tag{3.17}$$

$PO(t,S_t;p,K,T)$ denotes the t-price of the T-payoff $(K-S_T^p)^+$, i.e. the price of a power-put on S with strike K and power p. In particular,

$$PO(t,S_t;p,K,T) = e^{-r(T-t)}\left[K\mathcal{N}\left(-h_2\left(t,\frac{S_t}{e^{-r(T-t)}\sqrt[p]{K}}\right)\right) - \left(\frac{S_t}{e^{-r(T-t)}}\right)^p e^{-\frac{1}{2}p(1-p)\sigma^2(T-t)}\mathcal{N}\left(-h_1\left(t,\frac{S_t}{e^{-r(T-t)}\sqrt[p]{K}}\right)-(1-p)\sigma\sqrt{(T-t)}\right)\right], \tag{3.18}$$

where $\mathscr{N}$ denotes the one–dimensional standard normal distribution function, and where the functions $h_{1,2}$ are given by

$$h_1(t,z) = \frac{\ln z + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{(T-t)}}; \quad h_2(t,z) = h_1(t,z) - \sigma\sqrt{(T-t)}.$$

Proof. First consider (i). With Equation (3.13) and the relation between the portfolio weights and the numbers of assets, it follows

$$\begin{aligned}\left[G - V_T^{B\&H}\right]^+ &= \left[G - \left(\phi_{0,S}^{B\&H} S_T + \phi_{0,D}^{B\&H}\right)\right]^+ \\ &= \phi_{0,S}^{B\&H}\left[K^{B\&H} - S_T\right]^+\end{aligned}$$

where $K^{B\&H} := \frac{G-\phi_{0,D}^{B\&H}}{\phi_{0,S}^{B\&H}}$. If $\phi_{0,D}^{B\&H} \geq G$, i.e. if the number of bonds with maturity T is larger than the guarantee, the put expires out of the money a.s. In this case, the buy and hold strategy is a static portfolio insurance strategy. For $\phi_{0,D}^{B\&H} < G$, the value of the embedded option is given by $\phi_{0,S}^{B\&H}$ times the value of a standard European put on the asset S with strike $K = K^{B\&H}$ which completes the proof.

Now consider (ii). With Equation (3.13), it follows

$$\begin{aligned}\left[G - V_T^{CM}\right]^+ &= \left[G - V_0^{CM} h_T(\pi_0)\left(\frac{S_T}{S_0}\right)^{\pi_0}\right]^+ \\ &= \frac{V_0^{CM} h_T(\pi_0)}{S_0^{\pi_0}}\left[K^{CM} - S_T^{\pi_0}\right]^+ = \vartheta\left[K^{CM} - S_T^{\pi_0}\right]^+,\end{aligned}$$

where $\vartheta := \frac{V_0^{CM} h_T(\pi_0)}{S_0^{\pi_0}}$ and $K^{CM} := \frac{G}{\vartheta}$. The price is thus given by ϑ times the t_0-value of a power option $PO(t_0, S_0; p, K, T)$ with power π_0 and strike price K^{CM}.

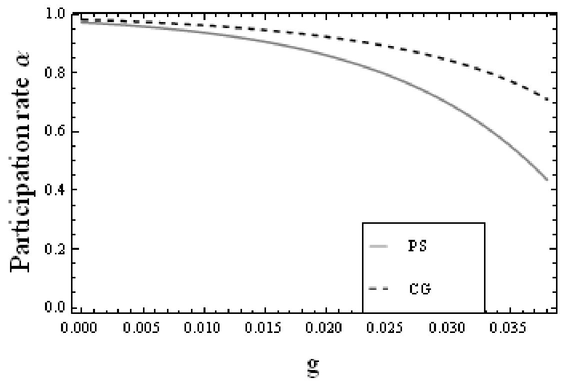
Finally consider (iii). With Equation (3.13), it follows that in a frictionless market $P\frac{V_T^{CPPI}}{V_0^{CPPI}} > G$ a.s.. The put option thus expires out of the money a.s. This implies that the guarantee has a value of zero, which in turn implies that the fair participation rate for all guarantee schemes is equal to one.

The pricing formula for the power claim is a well known result in literature, see for example Zhang (1998), p. 597, Equation (30.3). The proof is easily done by using a change of measure, c.f. Esser (2003) or Mahayni and Schlögl (2008). □

The interpretation is straightforward if one considers the payoffs V_T^b summarized in Lemma 3.1. In case of the buy and hold strategy, a positive investment into the risk-free asset results in a lower boundary for the value of the strategy at maturity. The amount taken care of by the guarantee is then only the difference between the level of the guarantee and this lower boundary. If the position in the bond is large enough, there is no need to protect the guarantee. For the constant mix strategy, on the other hand, the value of the portfolio can become arbitrarily low as soon as the portfolio weight is positive. The guarantee then always has some value. Finally, consider the CPPI strategy. In a frictionless market and for a continuous asset price process, the terminal value at T is at least as large as the guaranteed amount G with probability one. There is no need to add a guarantee. The combination of the CPPI with any guarantee scheme results in the CPPI itself.

3.3.4 Analysis and illustration of fair insurance payoffs

The numerical examples are based on the following benchmark parameters. The maturity of the insurance contracts is set to $T = 20$ years. The risk free interest rate is set equal to $r = 0.0455$, the volatility of the stock price is $\sigma = 0.15$. V_0 and S_0 are normalized to one. If not mentioned otherwise, the guaranteed rate is set equal to $g = 0.02$. First, the fair contract parameters for the guarantee schemes $w \in \{CG, PS\}$ are determined when the investment strategy is given by a pure investment into the stock, so that *B&H* and constant mix coincide. The left part of Figure 3.2 gives the fair participation rate α^* as a function of g. As pointed out in Section 3.2, the participation rate α^* is a decreasing function of the guaranteed interest rate g. It is larger (or equal) for the contribution guarantee scheme than for the participation surplus scheme. The right graph in Figure 3.2 illustrates the fair α^* for varying maturities T and the two guarantee schemes. The fair participation rate of the CG scheme is almost invariant with respect to the time horizon T, whereas the PS scheme results in a participation rate α^* which is monotonously increasing with maturity. To get the intuition, note that the level of $I(T,P)$ for which the investor participates in gains differs between CG and PS.



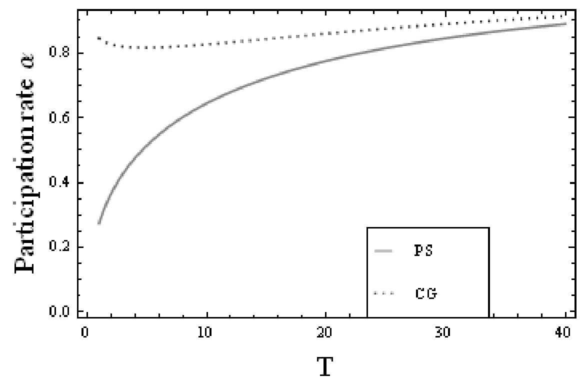


Fig. 3.2 Fair participation rates α for $\pi = 1$.
The left (right) graph shows the fair participation rate α^* for a pure stock investment with guarantee schemes $w \in \{CG, PS\}$ as a function of the guaranteed rate g (time to maturity T).

As can be seen from Figure 3.1, it is lower for PS than for CG. For a low time to maturity, the probability to end up below this level (and thus to need the insurance by the put) is thus significantly larger for PS. This implies a larger price of the embedded put option in case of PS, which in turn results in a lower participation rate. If the time to maturity goes to infinity, the probability decreases towards zero (assuming that $r > g$). The difference between the two schemes and thus between the two participation rates α^*_{CG} and α^*_{PS} vanishes.

Secondly, we analyze the fair contract parameters for different underlying investment strategies V_T^b with $b \in \{B\&H, CM, CPPI\}$. In case of a CPPI strategy, the fair participation rate is always equal to one independent of the guarantee scheme, since there is no gap risk in our model. For the other two strategies, Figure 3.3 gives the fair participation rate as a function of the initial weight of the stock π_0. The buy and hold strategy results in a static portfolio insurance for $\phi_{0,D}^{B\&H} \geq G$, i.e. here for $\pi_0^{B\&H} \leq 0.4$. In this cases, the put option is worthless so that the fair participation rate is $\alpha^*_{CG} = \alpha^*_{PS} = 1$.

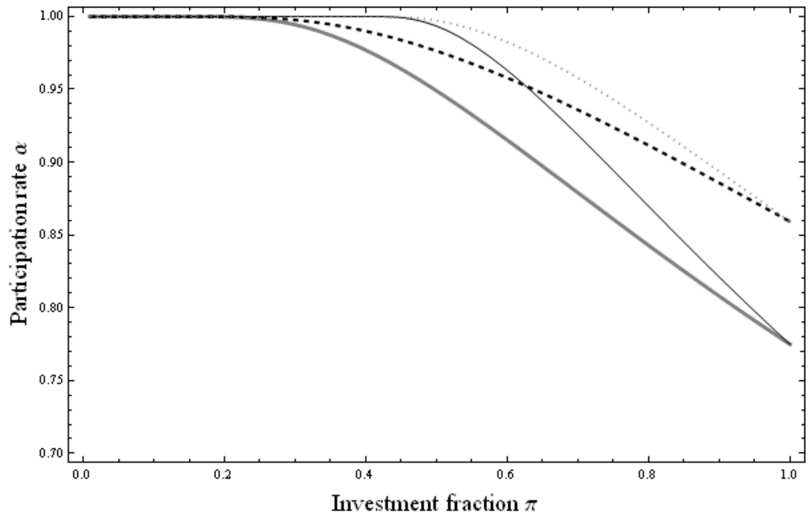


Fig. 3.3 Fair participation rates α for buy and hold and constant mix strategies.
The figure shows the fair participation rate α^* as a function of the initial investment fraction π_0 for the guarantee schemes CG (dotted line) and PS (solid line) and the constant mix strategy (thick lines) and the buy and hold strategy (thin lines).

3.4 Optimal insurance contract and benchmark utility

The insured has to decide simultaneously on the underlying self-financing investment strategy and on the guarantee scheme, where he is restricted to the fair contract specifications. We assume that the insured maximizes his expected utility, where his utility function is u. Since we are interested in the benefits derived from insurance contracts for savings towards retirement, we assume that the investor has utility from terminal wealth only. His optimization problem is[13]

$$\max_{\pi\in\Pi, w\in\{CG,PS\}} \mathbb{E}_{\mathbb{P}}\left[u(\mathscr{S}_w(I(T,P),\alpha_w^*,g_w^*)\right] \;\; s.t. \text{ Equations } (3.4),(3.12) \text{ and } (3.14), \tag{3.19}$$

where the restrictions ensure that the underlying investment strategy is self-financing and that the contract is fair. Π is the set of all predictable investment fractions $(\pi_t)_{0\leq t\leq T}$. The choice of the strategy $\pi\in\Pi$ defines the terminal value V_T and via Equation (3.14) also the terminal value $I(T,P)$ of the underlying investment strategy.

[13] Note that the expected utility has to be calculated under the real world measure $\mathbb{P}$, while the fair contract parameters have to be determined under the risk-neutral measure $\mathbb{P}^*$.

Theorem 3.1 (Optimal Insurance Contract with guaranteed interest rate). *For a CRRA-investor with utility function $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, the solution to the portfolio planning problem (3.19) is given by a constant mix strategy with*

$$\pi_t^{*,CM} = \frac{\mu - r}{\gamma\sigma^2} \quad \textit{for all } t \in [t, T]$$

and the guarantee scheme CG.

For a HARA investor with subsistence level $G = Pe^{gT}$ and utility function $u(x) = \frac{(x-G)^{1-\gamma}}{1-\gamma}$, the solution to the portfolio planning problem is given by a CPPI strategy with terminal floor G and multiplier $m^* = \frac{\mu - r}{\gamma\sigma^2}$, combined with any guarantee scheme.

Proof. For a CRRA investor the optimal solution of

$$\max_{\pi \in \Pi} \mathbb{E}_{\mathbb{P}}\left[u\left(V_T(\pi)\right)\right] \;\; s.t. \;\text{ Equation (3.12)}$$

is a CM strategy with

$$\pi_t^{*,CM} = \frac{\mu - r}{\gamma\sigma^2} \quad \text{for all } t \in [t, T]. \tag{3.20}$$

The solution to this problem is well known and goes back to Merton (1971).

Adding an exogenous restriction to this problem such that $V_T > G$ implies the following payoff function

$$V_T^R(\pi) = \max\left\{\tilde{V}_T(\pi), G\right\} = \tilde{V}_T(\pi) + \left[G - \tilde{V}_T(\pi)\right]^+$$

For $G = 0$ the solution reduces to the classical (unrestricted) Merton case. Iff $G > 0$, the optimal solution affords a reduction of the initial investment ($\tilde{V}_0 \leq V_0$) to finance the guarantee. El Karoui et al. (2005) show that the optimal solution is given by

$$V_T^R(\pi^*) = \tilde{V}_T^{CM}(\pi^{*,CM}) + \left[G - \tilde{V}_T^{CM}(\pi^{*,CM})\right]^+ \tag{3.21}$$

$$\text{where } \tilde{V}_T^{CM} = \underbrace{\tilde{V}_0 h_T(\pi^{*,CM}) \left(\frac{1}{S_0}\right)^{\pi^{*,CM}}}_{\tilde{\vartheta}} S_T^{\pi^{*,CM}}. \tag{3.22}$$

The parameter $\tilde{\vartheta}$ (from which we then get $\tilde{V}_0$) solves the budget restriction

$$V_0 = \tilde{V}_0 + \tilde{\vartheta} \cdot PO\left(t_0, S_0; \pi^{*,CM}, \frac{G}{\tilde{\vartheta}}, T\right), \tag{3.23}$$

where PO is defined in Proposition 3.2. Rearranging Equation (3.21) yields

$$V_T^R(\pi^*) = \frac{\tilde{V}_0}{V_0} V_T^{CM}(\pi^{*,CM}) + \left[G - \frac{\tilde{V}_0}{V_0} V_T^{CM}(\pi^{*,CM})\right]^+.$$

Comparing this expression with the *fair* payoff from a CG scheme and an underlying investment strategy from the CM–class

$$\mathscr{S}_{CG}(V_T^{CM}, \alpha^*, g^*) = \alpha^* V_T^{CM} + \left[G_T - \alpha^* V_T^{CM}\right]^+ \quad \text{with } V_0 = P$$

shows that the optimal payoff results from a CG scheme applied to a CM strategy with $\alpha^* = \frac{\tilde{V}_0}{V_0}$, $V_0 = P$ and $\pi^{CM} = \pi^{*,CM} = \frac{\mu - r}{\gamma\sigma^2}$.

For an investor with utility function $u(x) = \frac{(x-G)^{1-\gamma}}{1-\gamma}$, Basak (2002) has shown that the optimal solution is given by a CPPI strategy with multiplier

$$m^* = \frac{\mu - r}{\gamma\sigma^2}. \tag{3.24}$$

The terminal value of the CPPI strategy is at least as large as the guarantee level G. The payoff of a guarantee scheme based on the CPPI thus coincides with the payoff from the CPPI. This shows that any guarantee scheme, applied to the CPPI, is optimal, and it furthermore shows that the fair participation rate is equal to one. □

3.4.1 Utility loss

To assess the utility losses due to an exogenous guarantee, we restrict the analysis to a CRRA investor with no subsistence level.[14] We give the utility losses of the optimal restricted solution (where the restriction is the exogenous guarantee) and of fair suboptimal combinations of investment strategies and guarantee schemes. The comparison

[14] In case of a subsistence level, the investor applies the CRRA-utility function to the cushion exceeding the guarantee level. Thus, one can interpret our analysis such that we consider the utility loss due to a (too high) guarantee which exceeds the subsistence level by G.

is based on the certainty equivalents (CE) at time T. This certainty equivalent is the deterministic amount for which the investor is indifferent between getting this deterministic amount at T or using an insurance contract with underlying investment rule $\pi^b \in \Pi^b$ where $b \in \{B\&H, CM, CPPI\}$ and guarantee scheme $w \in \{CG, PS\}$:

$$u\left(CE_T(\pi^b, w)\right) = \mathbb{E}_{\mathbb{P}}\left[u\left(\mathscr{S}_w(\pi^b)\right)\right]$$

For a CRRA investor who does not face any restrictions, the optimal solution is given by a CM-strategy with investment fraction $\pi^{*,CM}$. The corresponding maximal certainty equivalent CE_T^*, which will serve as the benchmark in the following, is given by

$$CE_T^* = V_0 e^{\left(r+\pi^{*,CM}(\pi^{*,CM}-r)-\frac{1}{2}\gamma\pi^{2,*,CM}\sigma^2\right)T}.$$

The utility loss is then measured by the so-called loss rate l which gives the annualized loss in the certainty equivalent due to the use of a suboptimal insurance contract.

Definition 3.3 (Loss rate). The loss rate $l_T^{(\pi^b,w)}$ of the insurance contract with underlying investment rule $\pi^b \in \Pi^b$ and guarantee scheme w relative to the optimal solution CE_T^* over an investment horizon of T years is

$$l_T^{(\pi^b,w)} = \frac{\ln \frac{CE_T^*}{CE_T(\pi^b,w)}}{T}. \tag{3.25}$$

By means of the loss rate, we then compare different combinations of investment strategies and guarantee schemes for the same time horizon T.

3.4.2 Comparison of the utility losses

Our benchmark parameters are set as in subsection 3.4. In addition we need the risk aversion which is set to $\gamma = 1.7$, and the drift of the stock $\mu = 0.08$. In a first step, we analyze the utility losses due to choosing a suboptimal guarantee scheme. The optimal restricted solution is given by a CM strategy, combined with a contribution guarantee scheme (CG).

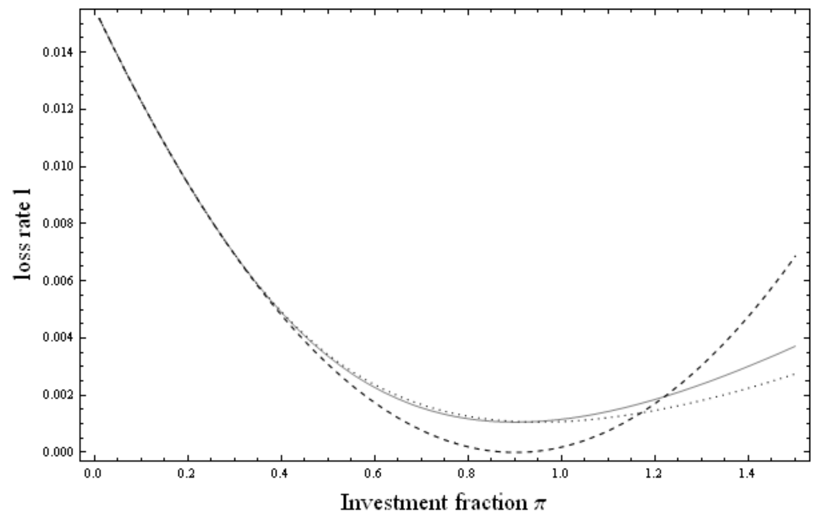


Fig. 3.4 Loss rate for constant mix strategy.
The figure gives the loss rate l of constant mix strategies without guarantee scheme (dashed line), with the guarantee scheme CG (solid line) and with the guarantee scheme PS (dotted line) as a function of the initial investment fraction π_0.

Figure 3.4 gives the loss rates if the investor uses a CM strategy with no guarantee (our benchmark), with the CG scheme or with the PS scheme as a function of the initial investment fraction. For a planning horizon of $T = 20$ years, the loss rate due to the exogenous guarantee is around 16 bp. Notice that the additional loss rate due to the use of the suboptimal PS scheme instead of the CG scheme can basically be ignored, given that the investment strategy is chosen optimally. The investor thus suffers more from the introduction of an exogenous guarantee than from being forced to use a suboptimal guarantee scheme. However, if the investment fraction π_0 is also given exogenously (instead of being chosen by the investor), the choice of the scheme can be crucial. These findings are also confirmed by Table 3.2, which gives the loss rates and the optimal investment fractions for various volatility levels of the stock, and for all combinations of strategies and guarantee schemes. In addition, it is illustrated that the ranking in terms of the utility loss for the combinations of investment strategies and guarantee schemes holds independent of the investment horizon.

For a B&H strategy and a CPPI (combined with either the CG or the PS scheme), the loss rates are shown in Figure 3.5. The losses due to the use of the suboptimal B&H strategy are very low.[15] The differences between the guarantee schemes can basically be ignored. The loss rates for the CPPI, which already meets the guarantee by construction, are larger. Using the CPPI instead of the CM strategy combined with a CG scheme leads to an additional loss rate of at least 50%. The figures show that – even

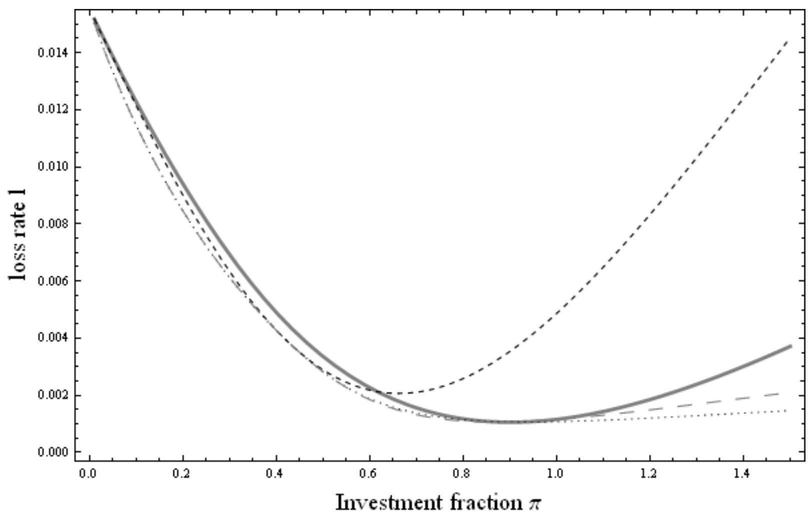


Fig. 3.5 Loss rates for buy-and-hold and CPPI.
The figure gives the loss rate l of the optimal insurance contract given by a constant mix strategy combined with a contribution guarantee scheme (thick solid line), a buy and hold strategy combined with CG (thin solid line) and PS (dotted line) and the CPPI strategy (dashed line) as a function of the initial investment fraction π_0.

if the loss rates are rather similar – the optimal investment fractions for the various strategies and guarantee schemes can be rather different. They also show that the use of a suboptimal π_0 can cause way larger losses than the use of a suboptimal investment strategy or guarantee scheme.

[15] This result is in line with the findings of Rogers (2001), who shows that the utility losses from discrete instead of continuous trading are very low for realistic parameter constellations.

	$T=20,\sigma=0.15$ $\pi^{*,CM}=0.90$		$T=10,\sigma=0.15$ $\pi^{*,CM}=0.90$		$T=20,\sigma=0.2$ $\pi^{*,CM}=0.51$	
	CG	PS	CG	PS	CG	PS
Constant Mix	0.00105 (0.90)	0.00106 (0.95)	0.00234 (0.90)	0.00235 (1)	0.00032 (0.51)	0.00033 (0.52)
Buy and Hold	0.00106 (0.87)	0.00106 (0.94)	0.00235 (0.86)	0.00235 (1)	0.00070 (0.48)	0.00070 (0.48)
CPPI	0.00206 (1.66)		0.00375 (2.38)		0.00091 (1.01)	

Table 3.2 Loss rates and optimal investment fractions.
The table gives the utility losses for different combinations of investment strategies and guarantee schemes. The optimal investment fraction in case of the constant mix and the buy-and-hold strategy and the optimal multiplier in case of the CPPI, respectively, are given in brackets.[16]

3.5 Conclusion

This chapter has analyzed which fair combination of a self-financing investment strategy and a guarantee scheme is the optimal choice for an insured who maximizes expected utility. In line with regulatory requirements, the terminal guarantee is imposed exogenously. We consider two basic guarantee schemes which are also applied in practice. The contribution guarantee scheme is in the spirit of guaranteed minimum accumulation benefits belonging to the class of variable annuities. The participating guarantee scheme can be found within equity–linked contracts. The set of investment strategies is not restricted besides the self–financing condition.

For each combination of an investment strategy and a guarantee scheme, we determine the fair (arbitrage–free) combinations of guaranteed rate and participation rate. This results in the set of all feasible contracts the insured can choose from. We show that a constant mix strategy combined with the contribution guarantee scheme is optimal for a CRRA investor. An investor who has a subsistence level optimally chooses a CPPI strategy which meets the guarantee by construction.

We emphasize the crucial interdependence between guarantee scheme and the underlying investment strategy. The insured has to decide on these two components simultaneously. We also analyze utility losses which result from choosing suboptimal investment strategies, suboptimal guarantee schemes, or both. It turns out that the difference between the guarantee schemes can basically be ignored. However, the loss rates for the CPPI strategy can be significant.

Chapter 4

Variable Annuities and the Option to seek risk: Why should you diversify?[1]

4.1 Introduction

This chapter focuses on guaranteed minimum accumulation benefits. Here, the insured can choose between products including a rider to (dynamically) decide on the investment himself and contracts where the provider decides on the investment strategy.

Recently, these contracts have been launched with an additional option component, the rider to shift/switch between multiple funds.[2] Compared to the products without the rider to switch, the investor gains flexibility on her investment decisions. However, this does not come for free. The provider does not know the investment decisions a priori.[3] We reason that he will make sure that, no matter which investment decisions the insured chooses, he is able to hedge the risk of the guarantee option. If the investor deviates from the *worst case* investment strategy, she pays too much for her guarantee. Our main focus is on the incentive effects which are caused by the additional rider. We compare the optimal investment decisions with and without the additional rider to switch and analyze the utility gain/loss which is caused by the new product design.

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[1] This chapter is based on joined work with Antje Mahayni.

[2] Products traded recently on the market are AXA Twinstar, Allianz Invest4Life, R+V Premium GarantRente and Swiss Life Champion. To be more precise, switching refers to the case where the investor can only decide how to invest the ongoing premia whereas shifting refers to the entire account value.

[3] Stated differently, the insurance faces the problem of asymmetric information and moral hazard of the investor.

Although the main motivation of this chapter stems from guaranteed minimum accumulation benefits, the results are valid for all products where the underlying of a put–option is an investment strategy. We use some assumptions which simplify the exposition but do not alter the core of our statements. First of all, we restrict ourselves to a single up front premium instead of the more general version of periodic premia.[4] We use a multi asset Black–Scholes model and assume that the entire account value can be rearranged continuously.[5] The optimal strategies are derived by maximizing expected CRRA–utility of terminal wealth where the investor also obtains utility from an independent non-market wealth.

The main contributions of the chapter are as follows. We identify the *worst case* strategy, i.e. the strategy which gives the highest value for the guarantee (guarantee put option, respectively) as well as the utility maximizing strategy.[6] The optimal unrestricted (diversified) strategy can be implemented into a GMAB contract if and only if the premium and the strategy are determined simultaneously. This implies that the investor must commit herself to a strategy at the contract inception. Obviously, this is not in the spirit of the additional rider to switch. The time lag between the premium payment and the investment decisions causes a dilemma for the investor. On one hand, she still benefits from diversification. On the other hand, her guarantee is more valuable in the case of a more aggressive strategy. In consequence her optimal investment strategy for a contract which includes the rider to switch mitigates between the *optimal*

[4] This introduces an Asian option feature and impedes closed-form solutions. VA's are originated from the US where single up-front premiums are standard. In Germany most products are offered with ongoing premiums. The pricing of Asian options is extensively discussed in the literature. Without postulating completeness, we refer to the works of Curran (1994), Rogers and Shi (1995), Nielsen and Sandmann (2002b) or Chen et al. (2008).

[5] In practice, rearranging the account is normally allowed at most four times per year. This gives rise to an interesting stopping problem which is beyond the scope of this chapter. Mahayni and Schoenmakers (2011) analyze the optimal stopping problem in the case of one switching right within different model classes.

[6] A discussion how the riskiness of assets (distributions) is measured already dates back to Rothschild and Stiglitz (1970). The relation between the value of a call–option and the riskiness of its underlying stock is firstly discussed in Jaganathan (1984).

diversified and the *worst case* (most risky in terms of the put price) strategy.[7] We show that the additional rider to switch causes a utility loss. Surprisingly, it turns out that the utility loss is not necessarily increasing in the level of risk aversion due to the borrowing constraints which are posed on the private investor. However, the utility loss is increasing after a critical level of risk aversion which is linked to the combined effects of borrowing constraints and background risk.

This chapter is related to several strands of the literature. First we like to mention some papers on utility losses/gains caused by guarantees. Jensen and Sørensen (2002) analyze wealth losses for pension funds and emphasize that the individual investor can substantially suffer from the investment strategy conducted by the sponsor. In particular, the losses for less risk averse investors due to the sponsor's deviation to less risky strategies than optimal can be quite pronounced. More recently, utility losses caused by guarantees are also analyzed in Balder and Mahayni (2010) and Branger et al. (2010). Branger et al. (2010) particularly focus on the link between utility loss and the investment strategy underlying a put option embedded into Variable Annuities. In contrast, Døskeland and Nordahl (2008) are able to explain the merits of guarantees by means of cumulative prospect theory.

Related literature also includes papers on the pricing of insurance contracts with guarantees and portfolio planning. There is extensive literature which deals with the fair pricing of insurance contracts with embedded options. Pricing embedded options by no arbitrage already dates back to Brennan and Schwartz (1976). A description of the different product groups which are subsumed under the class of variable annuities and a universal pricing framework is given in Bauer et al. (2008).

We also refer to the literature on Passport options, cf. Andersen et al. (1998), Henderson and Hobson (1998), Delbaen and Yor (1998), Nagayama (1999), and Shreve and Vecer (2000). To some extent, these options are similar to our problem formulation. Passport options are options on an investment strategy (traded account, respectively), too. Their pricing also relies on the value maximizing strategy. Nevertheless,

[7] In fact, as pointed out especially in Milevsky and Kyrychenko (2008), policyholder which select additional rider, indeed take on more financial risk.

the pricing problem is different to ours. The strategies underlying a Passport option are restricted by their number of assets. In contrast, the (implicit) restriction which the investor of a guaranteed minimum accumulation benefit faces concerns the fractions of the account value which she can invest in multiple assets, i.e. we have *limits on the investment fractions*.

Literature on portfolio planning, in particular with an emphasis on insurance contracts with guarantees, includes, Huang et al. (2008), Milevsky and Kyrychenko (2008) and Boyle and Tian (2008). Portfolio planning itself dates back to Merton (1971) who, amongst other results, solves the portfolio planning problem for a CRRA investor. It is well known in the literature that including a background asset like non market wealth or labor income, can have significant impact on the asset allocation. Bodie et al. (1992) point out that a non-stochastic income stream where the investor can choose the labor supply in each period substitutes for bond investments. An overview of early works on optimal policies with deterministic income is given in Svensson and Werner (1993). Cocco (2005) shows that an investor having additional housing assets can conduct a riskier strategy in the market related assets. In the presence of stochastic positive labor income where the correlation is insignificant positive or zero similar effects are pointed out by Koo (1998), Viceira (2001) and Polkovnichenko (2007). However, substantial negative shocks in labor income can result in a more conservative asset allocation strategy as pointed out in Cocco et al. (2005). Kimball (1993), Gollier (1996), Eeckhoudt et al. (1996), Franke et al. (1998) and Franke et al. (2011) focus on the investor's derived risk aversion in the presence of background risks. Franke et al. (2011) show that for a positive additive background risk with a moderate volatility the investor becomes less risk averse at each point in time and the derived risk aversion is similar to the one of a HARA investor.

The rest of the chapter is organized as follows. In Sec. 2, we define the contract under consideration. We emphasize on admissible investment strategies and the decomposition of the up–front premium into an investment and a hedging part. Furthermore, we differentiate between fair and feasible contracts, i.e. contracts with and without the additional rider to switch. In Sec. 3 we turn to the optimal choice of investment strategies for the GMAB in the presence of background risk. We show that the rider to switch implies an incentive to invest more riskily in terms of the guarantee costs. We also show that a GMAB contract with switching right gives rise to a lower expected utility than the otherwise identical contract without the rider. For varying levels of risk aversion, we illustrate the utility losses by means of realistic examples in Sec 4. Sec. 5 concludes the chapter.

4.2 Contract specification, model setup and pricing

For a given maturity $T > 0$, the payoff of a *GMAB* is given by the maximum of some guaranteed value G_T and the account value V_T, i.e.

$$GMAB_T := \max\{G_T, V_T\} = G_T + [V_T - G_T]^+ = V_T + [G_T - V_T]^+ .$$

Thus, the payoff can be decomposed into the guaranteed amount G_T and the payoff of a European call–option with strike G_T. Alternatively, it is given in terms of the account value V_T and the payoff of the corresponding put–option. If the contributions of the insured are given by an up–front premium P, the contract is called fair if the (arbitrage–free) price $GMAB_0$ is equal to the net premium P. Assuming a constant interest rate r and a fixed pricing measure $\mathbb{P}^*$ implies

$$P = \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}\left(V_T + [G_T - V_T]^+\right)\right] = V_0 + \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}\left[G_T - V_T\right]^+\right]$$

where $V_0 := \alpha P$ ($\alpha \in [0,1]$) denotes the initial account value and $G_T := Pe^{gT}$ $(g < r)$. Thus, w.l.o.g. we can set $P = 1$. A contract is called fair if

$$1 - \alpha = \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}\left[e^{gT} - V_T\right]^+\right]. \tag{4.1}$$

The fraction $(1 - \alpha)$ is called the hedging fraction. Accordingly, α is called the investment fraction.

Now, we consider the account value. The account value V is linked to an investment strategy φ in $N+1$ assets $S_0,\ldots,S_N$.[8] All stochastic processes are defined on an underlying stochastic basis $(\Omega,\mathscr{F},\mathscr{F}=(\mathscr{F}_t)_{t\in[0,T^*]},\mathbb{P})$ which satisfies the usual hypotheses. Trading terminates at time $T^*>T$. S_0 denotes a risk–free asset growing at the constant interest rate r ($r\geq 0$). S_i ($i=1\ldots N$) are risky assets. Throughout the following we assume[9]

$$dS_{t,0}=S_{t,0}r\,dt,\ \ S_{0,0}=s_0 \tag{4.2}$$

$$dS_{t,i}=S_{t,i}\left(\mu_i\,dt+\sum_{j=1}^{N}b_{ij}\,dW_{t,j}\right),\ \ S_{0,i}=s_i \text{ for } i=1,\ldots N. \tag{4.3}$$

$W=(W_{t,1},\ldots,W_{t,N})_{0\leq t\leq T}$ denotes a standard N–dimensional Brownian motion under the *real world* measure $\mathbb{P}$. μ_i ($\mu_i>r\geq 0$) and b_{ij} are constant. We use the convention $\sigma_i^2:=\sum_{j=1}^{N}b_{ij}^2$ to denote the quadratic variation of asset i and $\sigma_{ij}:=\sum_{k=1}^{N}b_{ik}b_{jk}$ for the covariation of asset i and j. A trading strategy is a predictable process $\varphi=(\varphi_0,\ldots,\varphi_N)$ with value process $(V_t(\varphi))_{t\in[0,T]}$ where $V_t(\varphi):=\sum_{i=0}^{N}\varphi_{t,i}S_{t,i}$. W.r.t. a GMAB contract, we call a strategy admissible investment strategy if it satisfies the following three conditions: (i)$\varphi_{t,i}\geq 0$ for all $i\in\{0,\ldots N\}$ and for all $t\in[0,T]$, (ii) $V_t(\varphi)=V_0(\varphi)+I_t(\varphi)$ where $I_t(\varphi):=\sum_{i=0}^{N}\int_0^t\varphi_{u,i}\,dS_{u,i}$ and (iii) $V_0(\varphi)=\alpha P$. (i) prohibits any short positions in the assets, (ii) is the self–financing condition, i.e. no money can be injected or withdrawn from the portfolio after an initial investment and (iii) gives the budget restriction. In summary, the value of the investment component at T is $V_T(\varphi)$ where φ is an admissible investment strategy. Assuming that continuous–time trading is possible, the financial market model described by Equations (4.2) and (6.1) is complete, i.e. there exists a uniquely defined martingale measure $\mathbb{P}^*$ such that W^* is a $\mathbb{P}^*$–Brownian motion and

[8] The investment opportunities can be interpreted as different mutual funds. In the following, we simply refer to assets.

[9] Notice that the main results of this chapter do not depend on the assumptions about the price dynamics. However, in this simplified model setup is convenient because it yields closed-form solutions for option prices and optimization problems.

$$dS_{t,i} = S_{t,i}\left(r\,dt + \sum_{j=1}^{N} b_{ij}\,dW^*_{t,j}\right), \;\; S_{0,i} = s_i. \tag{4.4}$$

Assume first that the investment strategy φ is known to the provider of the *GMAB* a priori. In this case, the price of the embedded put–option (the amount which is needed to hedge the put, respectively) is indeed given by the right hand side of Equation (4.1). For example, if the investment strategy implies a constant portfolio volatility $\sigma_{V,t} = \sigma$, the t–price of a European put–option with underlying V, maturity T and strike K is given by the well known formula[10]

$$\begin{aligned}&\mathscr{B}^{\text{Put}}(V_t, t, r, K, T)\\ &= -V_t\mathscr{N}\left(-d^{(1)}\left(t, \frac{V_t}{e^{-r(T-t)}K}\right)\right) + e^{-r(T-t)}K\mathscr{N}\left(-d^{(2)}\left(t, \frac{V_t}{e^{-r(T-t)}K}\right)\right)\end{aligned} \tag{4.5}$$

where $\mathscr{N}(.)$ denotes the one–dimensional cumulative distribution function of the standard normal distribution and

$$d^{(1)}(t,z) := \frac{\ln z + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}, \, d^{(2)}(t,z) := d^{(1)}(t,z) - \sigma\sqrt{T-t}. \tag{4.6}$$

The fair investment fraction $\alpha^{*,\text{fair}} = \alpha^*(g,\sigma)$ satisfying Equation (4.1) is implicitly given by[11]

$$\alpha^*(g,\sigma) = \frac{1 - e^{(g-r)T}\mathscr{N}\left(-\frac{\ln\alpha^*(g,\sigma) + (r-g)T - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)}{\mathscr{N}\left(\frac{\ln\alpha^*(g,\sigma) + (r-g)T + \frac{1}{2}\sigma^2\sqrt{T}}{\sigma\sqrt{T}}\right)}. \tag{4.7}$$

However, if the insured is allowed to determine the investment decisions dynamically, the provider does not know the strategy φ a priori. Here, the provider prices the embedded put according to the investment strategy which gives the highest guarantee value (embedded put value, respectively). The strategy $\bar{\varphi}$ defined by

$$\bar{\varphi} := \arg\sup_{\varphi} \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}\left[e^{gT} - V_T(\varphi)\right]^+\right] \text{ s.t. } \varphi \text{ is admissible} \tag{4.8}$$

[10] Notice that σ_V is constant if and only if the portfolio weights are constant.

[11] Notice that $g < r$ implies $\alpha^{*,\text{fair}} \in [0,1]$. In a more general model and contract setup, a detailed analysis of the existence of $\alpha^{*,\text{fair}}$ is given in Nielsen et al. (2009).

is called worst case strategy. In contrast to the fair investment fraction $\alpha^{*,\mathrm{fair}}$ implied by Equation (4.1), the investment fraction α^{wc} implied by the condition

$$1-\alpha = \sup_{\varphi} E_{\mathbb{P}^*}\left[e^{-rT}\left[e^{gT}-V_T(\varphi)\right]^+\right] \text{ s.t. } \varphi \text{ is admissible} \tag{4.9}$$

is called feasible investment fraction.

Proposition 4.1 (Worst case strategy, feasible investment fraction).
The worst case strategy $\bar{\varphi}$ is given by

$$\bar{\varphi}_{t,0} = 0 \text{ and } \bar{\varphi}_{t,i} = \frac{\alpha}{S_{0,i}} 1_{\{\sigma_i=\bar{\sigma}\}} \text{ for } i = 1,\dots N \tag{4.10}$$

where $\bar{\sigma} := \max_{i\in\{1,\dots,N\}} \sigma_i$. The feasible investment fraction α^{wc} is given by $\alpha^{wc} = \alpha^*(g,\bar{\sigma})$ where $\alpha^*(g,\sigma)$ is defined by Equation (4.7). In particular, it holds $\alpha^{wc} \leq \alpha^{*,fair}$.

Proof. Notice that an admissible strategy φ is self–financing and does not allow for any short positions. This implies that the (one–dimensional version) of the diffusion coefficient σ_V of the value processes $V_t(\varphi)$ is bounded from above by a constant volatility $\bar{\sigma}$ where $\bar{\sigma} = \max_{i\in\{1,\dots,N\}} \sigma_i$. The remainder of the proof is based on the robustness result of the Black/Scholes model (for convex payoffs) which is firstly derived in Avellaneda et al. (1995) and Lyons (1995).[12] If the diffusion coefficient σ_t of the underlying is bounded from above by a constant $\bar{\sigma}$, an upper price bound is implied by the Black/Scholes price according to $\bar{\sigma}$. In addition, the price bound of the embedded put is tight in the sense that it is achieved by the (admissible) strategy defined by Equation (4.10). □

It is worth mentioning that the above proposition is not only true with respect to strategies where the value process is lognormal, i.e. the volatility is deterministic. Instead, it is true for all admissible strategies. The robustness result which underlies the proof of the above proposition is stronger than the monotonicity of the Black/Scholes

[12] Detailed versions of the robustness result are also given in El Karoui et al. (1998), Dudenhausen et al. (1998) and Romagnoli and Varigiolu (2000).

Model and contract parameters

	Fund 1	Fund 2	Interest rate and correlation		Contract parameters	
μ_i	0.10	0.08	r	0.039	g	0.00
σ_i	0.29	0.15	$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$	-0.26	T	15

Table 4.1 Benchmark model and contract parameters.

pricing formula in σ. The last one gives the result w.r.t. the set of constant mix strategies, only.[13] In addition, there is a simple intuition behind the result. Any diversification between the assets reduces the risk. Thus, the worst case strategy is a single asset (non–diversifying) strategy where 100% of the risk capital is invested in the most risky asset. In a Black/Scholes type model setup, the most risky asset is given by the one with the highest volatility $\bar{\sigma}$.

The above results are illustrated for a *GMAB* with maturity $T = 15$, varying guarantee rates g and two (traded) mutual funds, the Invesco Bond A (Fund 1) and the UBS EF Global Opportunity (Fund 2).[14] The interest rate $r = 0.039$ is obtained from the Euroswap curve. An estimation of the Black/Scholes model is based on daily returns from April, 14, 2008 to April,7, 2010. The benchmark parameters which are also used in the remainder of the chapter are summarized in Table 4.1.

We consider an investor who equally splits the investment capital between the two fonds such that $\sigma_V = \frac{1}{2}\sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_{12}}$. In contrast, $\bar{\sigma} = \max\{\sigma_1, \sigma_2\}$. For varying guarantee rates g, the fair and feasible investment fractions $\alpha^{*,\text{fair}}$ and α^{wc} are illustrated in Figure 4.1. The reduction of the risk capital $\alpha^{*,\text{fair}} - \alpha^{\text{wc}}$ is only zero in the degenerated case that $g = r$ where the investment fractions are zero. The largest reduction of risk capital is observed for $g = 0.035$. Here, the feasible contract implies an

[13] For example, stopping strategies where the whole account value is switched from one asset to another are also admissible. In this case, the associated price of the GMAB is not given by a Black/Scholes pricing formula. However, using the robustness result, this price is still bounded by the Black/Scholes price at the upper volatility bound $\bar{\sigma}$.

[14] For example, a recently offered VA which is based on the above fonds is the SwissLife Champion provided by Swiss Life.

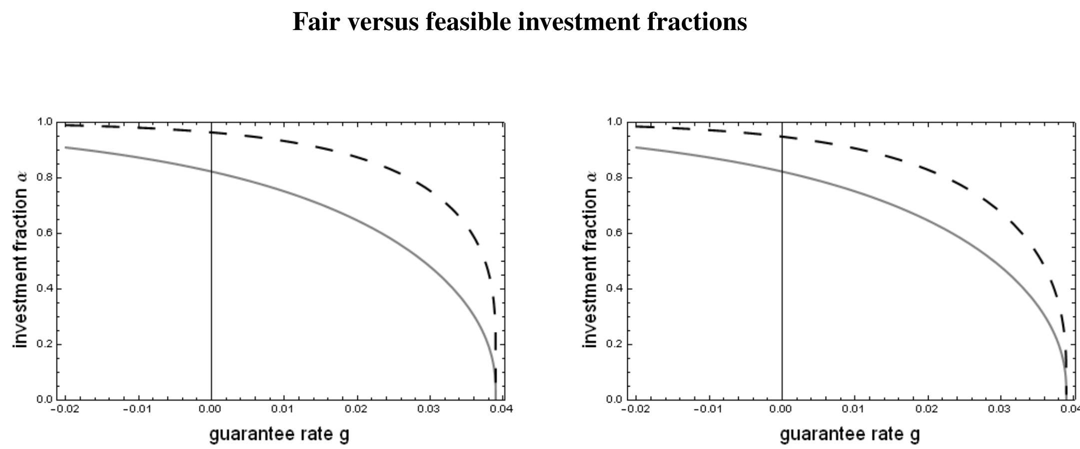

Fig. 4.1 Fair versus feasible investment fractions
The dashed lines refer to the fair investment fractions α^* according to an equal split between the fonds, the solid lines indicate the feasible fractions α^{wc}. While the left figure relies on the benchmark correlation $\rho_{12} = -0.26$ (obtained by the data), the right figure is based on a higher (in particular positive) correlation of $\rho = 0.26$.

investment fraction which is even 20% lower than the fair one. Obviously, the reduction is decreasing in the correlation of the assets.

4.3 Optimal choice of investment strategy under background risk

The worst–case strategy of Proposition 4.1 is a purely non–diversifying strategy in a single–asset. In general, a risk averse investor benefits from diversification. In view of the worst case price setting of the GMAB provider it is intuitive that the insured gets an incentive towards a more aggressive investment strategy compared to a contract without the additional rider. We consider an expected utility (of terminal wealth) maximizing investor with utility function $u(w)$ where $u'(w) > 0$ and $u''(w) < 0$. Solutions for the optimal investment decisions are then derived for the special case of a CRRA utility function $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$ for $w > 0$ and $u(w) = -\infty$ for $w \leq 0$. We take into account the existence of background risk. Along the lines of Franke et al. (2011), we assume a terminal background wealth X_T^B which is only resolved at the horizon date T and is independent of the GMAB–payoff.[15] There exists a variety of sources for X_T^B such as bequests, property sales, labor income or sales of shares in private businesses.[16] The terminal wealth TW_T of the investor is composed of her endogenous wealth $GMAB_T$ supplemented by the non–market, exogenous terminal (background) wealth X_T^B, i.e. $TW_T = X_T^B + GMAB_T$.

For a fair contract, the optimization problem relevant for the investor is

$$\max_{\varphi \in \Phi} \mathbb{E}_{\mathbb{P}} \left[u \left(X_T^B + GMAB_T(\alpha, g) \right) \right] \text{ s.t. } \alpha \text{ satisfies Equation } (4.1) \tag{4.11}$$

where Φ denotes the set of self–financing strategies with $V_0(\phi) = \alpha$. For a feasible contract, the relevant optimization problem is

$$\max_{\varphi \in \Phi} \mathbb{E}_{\mathbb{P}} \left[u \left(X_T^B + GMAB_T(\alpha^{\text{wc}}, g) \right) \right] \text{ s.t. } \alpha^{\text{wc}} \text{ satisfies Equation } (4.9). \tag{4.12}$$

In particular, along the lines of Proposition 4.1 we may set $\alpha^{\text{wc}} = \alpha(g, \bar{\sigma})$ where $\alpha(g, \bar{\sigma})$ is defined as in Equation (4.7) and $\bar{\sigma} = \max_{i \in \{1, \ldots N\}} \sigma_i$.

The contract parameter α is endogenous in the optimization problem (4.11), it is exogenous w.r.t. the optimization problem (4.12). A fair contract ensures that the risk capital α consists of the remaining part which is not needed to hedge the embedded put option. In contrast, for a feasible contract the risk capital is not linked to the *true* investment decisions. It is also worth to emphasize that the introduction of background risk is not innocent. Although the financial market is complete, the investor is not able to achieve any state dependent terminal wealth under a stochastic background risk. Thus, taking into account a (stochastic) background risk introduces a market incompleteness.

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[15] To be more precise, we assume that the background wealth X_T^B and the terminal asset prices $S_{T,i}$ ($i = 0, 1, \ldots, N$) are independent. Cocco et al. (2005) indeed estimate an insignificant positive correlation for their model.

[16] For a detailed overview, we refer to Campbell and Viceira (2002). For the sake of simplicity, we implicitly assume that the investor cannot influence the amount of X_T^B.

4.3.1 Comparison of put prices

In the first instance, we consider the effects on the hedging costs of a fair and a feasible contract, i.e. we compare the prices of put options where the underlying is the optimal investment strategy of the buyer of a fair and feasible contract. In the case of a fair contract, the interpretation of the put price as hedging costs is obvious. In the case of the feasible contract, the put price stemming from the optimized strategy can be interpreted as the hedging costs of a provider who anticipates that the insured maximizes her expected utility. Alternatively, one can interpret the difference of the worst case put price and the put price as the expected discounted value (under the martingale measure) of the outflows from a superhedge which is achieved by hedging the worst case put.

Theorem 4.1 (Comparison of put prices, utility loss). *Consider an investor described by a monotonously increasing utility function u, i.e. $u' > 0$ and terminal background risk X_T^B. Consider the set of random variables (account values) V_T which can be obtained by a self–financing strategy with borrowing constraints. Let $V_T^{*,fair}$ and $V_T^{*,feasible}$ solve*

$$\begin{aligned} V_T^{*,fair} &:= argmax_{V_T} \mathbb{E}_{\mathbb{P}} \left[u\left(X_T^B + V_T + [G_T - V_T) \right]^+ \right] \\ s.t.\ 1 &= \mathbb{E}_{\mathbb{P}^*} \left[e^{-rT} V_T \right] + \mathbb{E}_{\mathbb{P}^*} \left[e^{-rT} [G_T - V_T]^+ \right] \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} V_T^{*,feasible} &:= argmax_{V_T} \mathbb{E}_{\mathbb{P}} \left[u\left(X_T^B + V_T + [G_T - V_T) \right]^+ \right] \\ s.t.\ \alpha^{wc} &= \mathbb{E}_{\mathbb{P}^*} \left[e^{-rT} V_T \right]. \end{aligned} \tag{4.14}$$

It holds:

(i) The put option written on the payoff $V_T^{*,fair}$ is bounded from above by the price of the otherwise corresponding put option on $V_T^{*,feasible}$, i.e.

$$\mathbb{E}_{\mathbb{P}^*} \left[e^{-rT} [G_T - V_T^{*,fair}]^+ \right] \leq \mathbb{E}_{\mathbb{P}^*} \left[e^{-rT} [G_T - V_T^{*,feasible}]^+ \right]. \tag{4.15}$$

(ii) The feasible contract implies a utility loss L_T where

$$L_T := EU\left(V_T^{*,fair}\right) - EU\left(V_T^{*,feasible}\right) \geq 0$$

and $EU(V_T) := \mathbb{E}_{\mathbb{P}}\left[u\left(X_T^B + \max\{G_T, V_T\}\right)\right]$.

Proof. The proof is given in the appendix, cf. Appendix A.1.1. □

Intuitively, the above theorem is obvious. In the first instance, the investor maximizes her expected utility w.r.t the same set of random variables V_T which can be obtained by self–financing strategies which honor the borrowing constraints. Thus, we have implicitly the constraint that the put values are below or equal to $1-\alpha^{\mathrm{wc}}$. In the case of the feasible contract, the budget constraint gives two disadvantages compared to the fair contract. Firstly, the optimization is restricted to account values where the initial value is equal (or below) α^{wc}. Secondly, due to the borrowing constraints, it is not possible to obtain a higher put value than $1-\alpha^{\mathrm{wc}}$, anyway. Therefore, the difference of the budget in the case of the fair contract (normalized to 1) and the implicit budget of a feasible contract with (admissible) account value V (equal to $\alpha^{wc} + \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}[G_T - V_T]^+\right] < 1$) defines the sunk costs which are implied by the price setting based on the worst case strategy.

Basically, the main intuition for the incentive to invest more aggressively can be gained from the following two observations. The sunk costs are decreasing in the price of the put option and the price setting of the provider for the protection against unwanted outcomes is independent of the actual conducted strategy of the insured. Thus, the sunk costs are minimized if the optimal investment strategy is close to the riskiness of the worst case strategy. Therefore, the additional rider to switch implies higher protection costs for the provider but lower (or equal) to the ones implied by the worst case strategy.

Part (ii) of the theorem is also obvious. Under the assumption that the investor herself can not obtain different account values by herself than the provider, she can only loose utility by the additional contract rider because of the price setting effect. She suffers from the rider.

It is worth mentioning that Theorem 4.1 is impeded if one drops the assumption that the investor can choose within the same set of investment strategies with and without the rider. For example, in the case that the fair contract is restricted to buy and hold strategies or any other subclass of self–financing strategies (with borrowing constraints). Obviously, if the restricted investment opportunities do not contain the overall optimal strategy, it is possible that the utility loss which is caused by the worst case price setting can be mitigated by the utility gain caused by larger (better) set of investment strategies.

However, our main focus is on the utility loss which is caused by the worst case price setting. Based on the assumptions underlying Theorem 4.1, the maximal utility w.r.t. a contract including the rider is bounded from above by the optimal utility without the rider. Intuitively, it is clear that the utility loss depends on the level of risk aversion as well as the background risk. It is to be expected that an investor who wants to stay close to the worst case strategy, anyway, does not suffer as much from the price setting as an investor who seeks more diversification. I.e. the utility loss is expected to increase in the level of risk aversion and the risk that the background value is below its expected value. Before we give some numerical illustration of the utility loss, we discuss the influence of the background risk in the subsequent subsection.

4.3.2 Background risk, generalized constant mix strategies and optimality

To simplify the exposition, we first define the class of so called generalized constant mix strategies. In a second step we discuss their optimality as well as the link to background risk. The main insights are gained without taking into account for borrowing constraints. We comment on them in a subsequent subsection.

Definition 4.1 (Generalized constant mix strategies). Let $C_t := V_t + e^{-r(T-t)}l$ where l denotes some constant and $V_0 > -e^{-rT}l$. A strategy with portfolio weights $\pi = (\pi_0, \ldots, \pi_N)$ given by

$$\pi_{0,t} = 1 - \sum_{i=1}^{N} \pi_{i,t} \text{ and } (\pi_{1,t}, \dots \pi_{N,t}) = \frac{C_t}{V_t}(m_1, \dots, m_N) \tag{4.16}$$

where m_i ($i = 1, \dots, N$) is constant is called generalized constant mix strategy ($GM_{l,m}$). The strategy parameters are l and $m = (m_1, \dots, m_N)$. In particular, $-l$ denotes the floor of the terminal value and m is called the multiplier of the strategy.

For $l = 0$, the generalized constant mix strategy is a simple constant mix strategy where the portfolio weights are constant. Otherwise, for $l \neq 0$, the strategy can be interpreted as a constant mix strategy with respect to the *cushion* process C, i.e. a strategy with constant weights w.r.t. C.[17]

Using well known results from the literature gives the following proposition.

Proposition 4.2 (Optimality of generalized constant mix strategies). *Consider an investor described by a CRRA utility function $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$ for $w > 0$ and $u(w) = -\infty$ for $w \leq 0$.*

(i) Let $\mu_m := r + \sum_{i=1}^{N} m_i(\mu_i - r)$ and $\sigma_m := \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} m_i m_j \sigma_{ij}}$, and let $V_T^{GM_{l,m}}$ denote the terminal value of a generalized constant mix strategy with parameters l and m. Then it holds

$$\ln\left(\frac{V_T^{GM_{l,m}} + l}{v_0 + e^{-rT} l}\right) \sim N\left((\mu_m - \frac{1}{2}\sigma_m^2)T, \sigma_m^2 T\right). \tag{4.17}$$

(ii) A generalized constant mix strategy with parameters

$$l = \bar{x} \text{ and } (m_1, \dots, m_N)' = \frac{\bar{\mu}' \Omega^{-1}}{\gamma} \tag{4.18}$$

is the optimal solution for the optimization problem

$$\pi^* = argmax_{\pi} \mathbb{E}_{\mathbb{P}}[u(\bar{x} + V_T(\pi))] \text{ s.t. } \mathbb{E}_{\mathbb{P}^*}[e^{-rT} V_T(\pi)] = V_0. \tag{4.19}$$

(iii) Adding the terminal constraint $V_T \geq G_T$ to Problem (4.19) does not change the optimal proportions π^* but reduces the initial investment from $V_0 = 1$ to $\bar{V}_0 = \alpha$

[17] For $l < 0$ and $m > 2$, the generalized constant mix strategy is also known as a constant proportion portfolio insurance strategy (CPPI) which is firstly introduced in Black and Jones (1987).

where

$$\alpha = e^{-rT}\mathbb{E}_{\mathbb{P}^*}\left[\bar{V}_T^* + [G_T - \bar{V}_T^*]^+\right].$$

Proof. The proof is given in the appendix, cf. Appendix A.1.2. □

In particular, part (ii) of the above proposition states that the optimal strategy under a non–stochastic background risk with terminal value $\bar{x}$ is a generalized constant mix strategy with level $l = \bar{x}$ and a multiplier m along the lines of the well known *Merton solution*. While the weights of the cushion process are constant, the portfolio weights are constant in the special case that $C_t = V_t$, i.e. $\bar{x} = 0$, only. For $\bar{x} < 0$, the cushion C is lower than the portfolio value V. Here we have a CPPI strategy. In contrast, for $\bar{x} > 0$, the investor can (with certainty) add the present value of his future background wealth to the portfolio wealth, i.e. the cushion C is larger than V. Notice that for a given multiplier m, a generalized constant mix strategy is the riskier the higher the level l is. In particular, the introduction of a positive background value makes the CRRA–investor more aggressive.[18] If not mentioned otherwise, we assume in the following that the utility function is a CRRA function.

The above proposition immediately gives the optimal investment strategy in the case of a fair *GMAB* and a non–stochastic background wealth $X_T^B = \bar{x}$ a.s..

Proposition 4.3 (Non–stochastic background wealth). *For $X_T^{(B)} = \bar{x}$, the optimal strategy w.r.t. Problem (4.11) is a generalized constant mix strategy where the strategy parameters l and m are given by Equation (4.18). The (reduced) initial investment $\bar{V}_0 = \alpha$ (cf. part (iii) of Proposition 4.2) is given by*

[18] This is well known in literature see, Cocco (2005), Merton (1992) and Franke et al. (2011), i.e. the CRRA–investor behaves as a HARA–investor.

$$\alpha(g,\sigma,\bar{x}) = 1_{\{\bar{x}\leq -e^{gT}\}} + \tilde{\alpha} 1_{\{\bar{x}> -e^{gT}\}}, \tag{4.20}$$

$$\tilde{\alpha} = \frac{e^{-rT}\bar{x}\mathcal{N}\left(-d^{(1)}\left(0,\frac{\tilde{\alpha}+\bar{x}e^{-rT}}{e^{(\tilde{g}-r)T}}\right)\right) + 1 - e^{(\tilde{g}-r)T}\mathcal{N}\left(-d^{(2)}\left(0,\frac{\tilde{\alpha}+\bar{x}e^{-rT}}{e^{(\tilde{g}-r)T}}\right)\right)}{\mathcal{N}\left(d^{(1)}\left(0,\frac{\tilde{\alpha}+e^{-rT}\bar{x}}{e^{(\tilde{g}-r)T}}\right)\right)}, \tag{4.21}$$

$$\tilde{g} = \frac{\ln(e^{gT}+\bar{x})}{T} \text{ and } \sigma = \sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} m_i m_j \sigma_{ij}}. \tag{4.22}$$

$d^{(1)}$ and $d^{(2)}$ are given by Equation (4.6) and $\tilde{g} := \frac{1}{T}\ln(e^{gT}+\bar{x})$ denotes a modified guarantee rate. In particular, for $\bar{x}=0$ we have $\alpha(g,\sigma,\bar{x}) = \alpha^*(g,\sigma)$ defined as in Equation (4.7). For $\bar{x}\leq -e^{gT} < 0$, the put which is embedded into the GMAB is worthless.

Proof. The proof is given in the appendix, cf. Appendix A.1.3.

□

Notice that similar reasonings to the ones given subsequently to Proposition 4.2 also imply that the optimal strategy of a CRRA investor who buys a fair *GMAB* is more (less) aggressive for $\bar{x}>0$ ($\bar{x}<0$) than in the case without background wealth ($\bar{x}=0$).

Assumption 4.2 (Stochastic Background Risk)**.** *We assume that the background risk is bounded from below. In addition, we assume that the present value of the investments is at least as large as the highest lower bound on the background wealth, i.e. that*

(A1) There exists a constant κ such that X_T is bounded from below by κ, i.e. $X_T \geq \kappa$ a.s..
(A2) Let $\underline{\kappa}$ denote the tightest lower bound such that (A1) is true, i.e. $P_{X^B}\left(X_T^B \leq \underline{\kappa}\right) = 0$ and $P_{X^B}\left(X_T^B \leq \underline{\kappa}+\varepsilon\right) > 0$ for all $\varepsilon > 0$. We now assume that $e^{rT}v_0 + \underline{\kappa} > 0$, i.e. $v_0 > -e^{-rT}\underline{\kappa}$.

It is straightforward to show that the above assumption is necessary to obtain

$$\max_{l,m} E_{\mathbb{P}}\left[u\left(X_T + V_T^{GM_{l,m}}\right)\right] > -\infty$$

for a CRRA utility function u. Thus, throughout the following, we assume that Assumption 4.2 holds.

Due to the market incompleteness stemming from a stochastic background risk, the optimal investment strategies, in particular the ones underlying a fair and feasible GMAB contract, are to be determined numerically.The optimization problem can be simplified if one optimizes within the class of generalized constant mix strategies and guarantees a straightforward comparison of different background risks.

Proposition 4.4 (Expected utility of terminal wealth). *Let $GMAB_T^{l,m}(\alpha,g)$ denote the payoff of a GMAB with maturity T, contract parameters (α,g) where the investment strategy is a generalized constant mix strategy with parameters l and m. Then, for independent background risk X_T^B (bounded below by $\underline{\kappa}$) with density f_{X^B}, the expected utility of the terminal wealth $TW_T = X_T^B + GMAB_T^{l,m}(\alpha,g)$ of a CRRA–investor is given by*

$$E_{\mathbb{P}}\left[u(TW_T)\right] = \int_{\underline{\kappa}}^{\infty} E_{\mathbb{P}}\left[u(x+GMAB_T^{l,m}(\alpha,g))\right] f_{X^B}(x)\,dx \tag{4.23}$$

where

$$\begin{aligned} E_{\mathbb{P}}\left[u(x+GMAB_T^{l,m}(\alpha,g))\right] =& \frac{1}{1-\gamma}\left[\int_{\frac{l+e^{gT}}{\alpha+e^{-rT}l}}^{\infty} \left(x-l+(\alpha+e^{-rT}l)y\right)^{1-\gamma} g(y)\,dy \right.\\ &\left. + \,(x+e^{gT})^{1-\gamma}\mathcal{N}\left(\frac{\ln\left(\frac{l+e^{gT}}{\alpha+e^{-rT}l}-(\mu_m-\frac{1}{2}\sigma_m^2)\right)}{\sigma_m\sqrt{T}}\right)\right]. \end{aligned} \tag{4.24}$$

g is the density of a lognormal variable Y with $\ln Y \sim N\left((\mu_m-\frac{1}{2}\sigma_m^2)T, \sigma_m^2 T\right)$ where μ_m and σ_m are defined as in Proposition 4.2 and $h^{(2)}$ is defined as in Equation (4.6).

Proof. The proof is given in the appendix, cf. Appendix A.3.1. □

4.3.3 Borrowing constraints

Binding borrowing constraints imply that the investor is not allowed to invest as aggressively as he would like to. Qualitatively, the borrowing constrains mainly absorb some of the riskiness of the unrestricted strategy. For the optimization problems, borrowing constraints have a severe impact. It is not possible to exclude all path–dependent strategies from the optimization. In consequence, the determination of an optimized strategy affords sophisticated numerical methods.

In contrast, without the introduction of borrowing constraints, the optimal strategies in a Black and Scholes model setup are path–independent, independent from the utility function involved.[19] The optimal investment decisions and the portfolio value only depend on the current prices. They do not depend on the path which reaches the prices. Obviously, this is also true in the case of a non–stochastic background risk $\bar{x}$. Basically, the constant background value alters the utility function from CRRA to HARA. The path–independence still applies in the case of an independent stochastic background risk which only resolves at maturity T. However, the borrowing constraints lead to a path–dependent optimal strategy, even in the case of a non–stochastic background risk. A CPPI (generalized constant mix) strategy is optimal w.r.t a HARA utility function without the introduction of borrowing constraints. A simple method to adjust the CPPI to borrowing constraints is to cap the resulting investment fractions. Although this seems to be a natural candidate, this is not necessarily the optimal solution to the problem, see Grossman and Villa (1992).

In the following, we also use the natural (but not necessarily overall optimal) candidates, i.e. we do not allow that $C_t > V_t$ and set $\tilde{C}_t = \min\{V_t, C_t\}$ ($l \leq 0$, respectively). In particular, for lognormal background wealth, the optimization is then simply due to the class of constant mix strategies.[20]

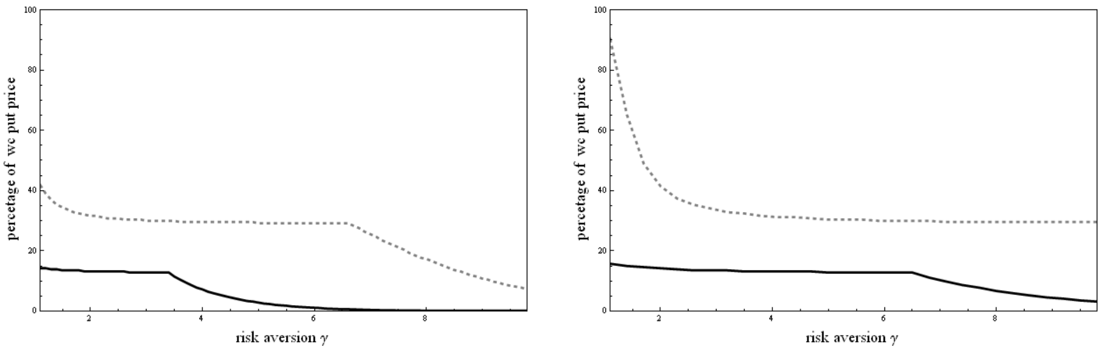
4.4 Numerical Illustration

If not mentioned otherwise, the following illustrations of the main statement, Theorem 4.1, are based on the benchmark setup summarized in Table 4.1. The utility function is

[19] See for example Cox and Leland (2000).

[20] Notice that the (optimal) adjustment to borrowing constraints is well known here. Basically, the adjustment is given by an artificial increase of the interest rate so that the borrowing constraints are met, cf. for example Tepla (2001).

Comparison of put prices for varying levels of risk aversion



Without background risk ($X_T^B = 0$) and a gurantee rate $g = 0.00$.

For lognormal background risk with $\mu_{X^B} = 0.05$ and $\sigma_{X^B} = 0.01$ and $X_0^B = 1$. ($g = 0.00$.)

Fig. 4.2 Fair (solid line) and feasible (dashed line) put prices as percentage of the worst case price for varying levels of risk aversion γ. In both plots, the time to maturity is $T = 10$. The other parameters are given as in Table 4.1.

a CRRA function, i.e. the preferences of the investor are exclusively specified by the level of risk aversion γ. A higher level of γ indicates a more risk averse investor. The background value X_T^B is assumed to be lognormally distributed, i.e.

$$\ln X_T^B \sim N\left(\ln X_0^B + (\mu_{X^B} - \frac{1}{2}\sigma_{X^B}^2)T, \sigma_{X^B}^2 T\right). \tag{4.25}$$

4.4.1 Comparison of put prices

We compare the prices of put options which are written on the optimal payoffs w.r.t a fair and the otherwise identical feasible contract, i.e. contracts without and with the additional rider to switch. The optimal payoffs are given by the payoff of a generalized constant mix strategy, i.e. a strategy parameterized by l and m. Under borrowing constraints, a lognormal background risk immediately implies that we can set $l = 0$.[21] For a fair contract, the optimal payoff maximizes Equation (4.23) where the input parameter α for Equation (4.24) is implicitly defined by the fair α according to the volatility

[21] Already Cocco et al. (2005) emphasize that because of moral hazard investors are prevented from capitalizing future labor (pension) income.

σ_m. In addition, the relevant values for μ_m are also determined by the volatility σ_m, i.e $\mu_m(\sigma_m)$ is the highest (efficient) μ_m for a given volatility level σ_m. Finally, the set of relevant σ_m is bounded because of the borrowing and short sale constraints. The procedure for a feasible contract is analogous. However, the input parameter α is not endogenous to the optimization problem. It is exogenously set equal to α^{wc}. For simplicity, we call the associated put options feasible and fair put.

Recall that Theorem 4.1, part (i), states that the price of the feasible put dominates the price of the fair one. Intuitively, it is clear that both put prices are decreasing in the level of risk aversion γ. This is illustrated in Figure 4.2 where, for varying γ, the put prices are plotted as percentage of the price of the worst case put option.[22] For $\gamma \to \infty$, the price of the feasible put converges to the price of the fair put. The put prices coincide if (and only if) the investor refrains from a risky investment. Thus, an equality of the put prices also implies that the puts are worthless. This is explained by the observation that the value of the optimal feasible payoff is lower than the value of the optimal fair payoff, i.e. $\alpha^{wc} < \alpha^{fair}$. In consequence, we only observes a feasible put price which is close to the fair put price if the risk aversion is high (and the background value is low).

While the left plot of Figure 4.2 refers to the case where the investor optimizes according to a zero background value $X_T^B = 0$, the right plot refers to a lognormal background value. Although the background value is not constant, we have $X_T^B > 0$ a.s. such that the direction of the impact is the same as in the case of a non–stochastic background value $\bar{x} > 0$.[23] The background risk makes the investor more aggressive. Thus, the put prices (percentage values) on the left hand side of Figure 4.2 which are calculated without background risk are lower than the ones in the right plot where the investor takes into account for a non–negative background value.

The effect is, ceteris paribus the larger (lower), the higher the background mean $\ln X_0^B + (\mu_{X^B} - \frac{1}{2}\sigma_{X^B}^2)T$ (standard deviation $\sigma_{X_B}\sqrt{T}$) is. This is illustrated in Figure 4.4

[22] For $T = 10$, $g = 0.00$ the value of a put written on a lognormal underlying with volatility $\sigma = \max\{\sigma_1, \sigma_2\} = 0.29$ is 0.2048.

[23] Cf. for example Franke et al. (2011).

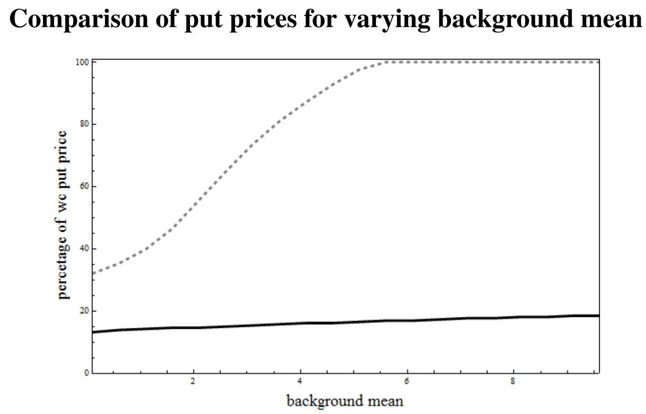


Fig. 4.3 Fair (solid line) and feasible (dashed line) put prices as percentage of the worst case price for varying mean of the background risk. The time to maturity is $T = 10$ and the risk aversion is set to $\gamma = 2$. The other parameters are given as in Table 4.1.

where the put prices (the percentage value of the worst case put price, respectively) are plotted for varying (increasing) means of the background value. Observe that both, fair and feasible put prices increase. However, the increase in the feasible put price is much higher compared to the increase in the fair put price. In particular, observe that the feasible put price is equal to the worst case put price for a mean background value of 6, the fair put price is still below 20% of the worst case put price. Intuitively, it is clear that the impact of the (positive) background risk is higher in the case of the feasible contract where the investor has an additional incentive to invest more aggressively. Without this incentive, a value close to the worst case put price is not observed for reasonable background means. We omit the illustration of the effect in the background volatility which is rather small. This is in line with the results of Cocco et al. (2005).

4.4.2 Utility loss

We give now an illustration of Theorem 4.1, part (ii), i.e. we consider the utility loss caused by the additional rider. We refer to the loss rates which are based on the certainty equivalents of the fair and the feasible contract. The certainty equivalent CE_T is the amount received at T for which the insured is indifferent to the random terminal wealth consisting of the payoff of the *GMAB*–contract and the background wealth X_T^B, i.e. CE_T is implicitly defined by

$$u(CE_T) = \mathbb{E}_{\mathbb{P}}\left[u\left(X_T^B + GMAB_T\right)\right].$$

For $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$ $(w > 0)$ it follows

$$CE_T = \left((1-\gamma)\mathbb{E}_{\mathbb{P}}\left[u\left(X_T^B + GMAB_T\right)\right]\right)^{\frac{1}{1-\gamma}}.$$

The annualized loss rate l_T^{rider} of the additional rider is given by

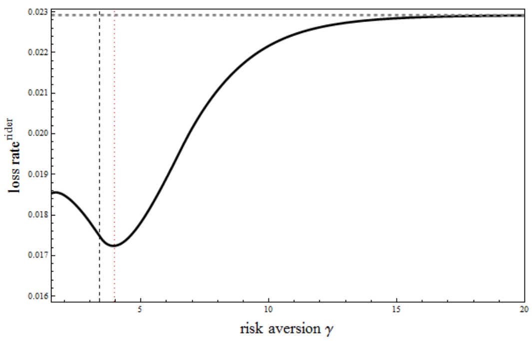
$$l_T^{\text{rider}} = \frac{\ln\left(\frac{CE_T^{*,\text{fair}}}{CE_T^{*,\text{feasible}}}\right)}{T} \tag{4.26}$$

where $CE_T^{*,w}$ denotes the certainty equivalent of the optimized contract $w \in \{\text{fair}, \text{feasible}\}$. Theoretically, the highest utility loss is observed if the investor refrains from any risky investments such that she does not need the insurance but still has to pay the worst case put price in the case of a feasible contract. Here, the feasible *GMAB*–payoff for a riskless investment strategy is given by $\max\{e^{rT}\alpha^{\text{wc}}, e^{gT}\}$ while the fair contract implies $\max\{e^{rT}, e^{gT}\} = e^{rT}$. Thus, for $X_T^B = 0$, we obtain a trivial upper bound for the loss rate $\bar{l}$ which is given by

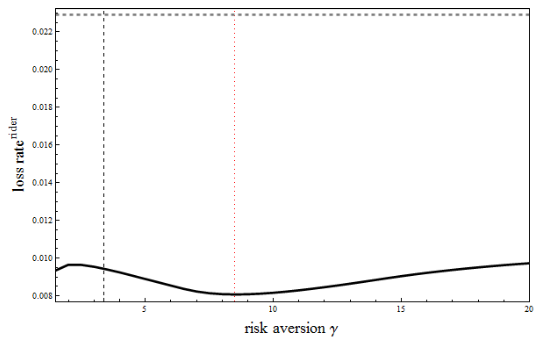
$$\bar{l} = \frac{1}{T}\ln\frac{e^{rT}}{\max\{e^{rT}\alpha^{\text{wc}}, e^{gT}\}} = -\frac{1}{T}\ln\max\{\alpha^{\text{wc}}, e^{(g-r)T}\}$$

Figure 4.4 illustrates the loss rate resulting in our benchmark scenario for varying levels of risk aversion γ. Observe that the loss rate is bounded from above by $\bar{l} = 0.023$. Intuitively, the upper bound is obtained in the case of an investor with sufficiently high risk aversion. While the upper bound is reached for $\gamma = 15$ without background risk, the loss rate is bounded from above by 0.01 in the case of a positive background value $X_T^B > 0$. Thus, the existence of a positive background risk can be viewed to mitigate the negative impact of the additional rider to some extent. This is in line with the observation that in this case the investor already invests more riskily and therefore the sunk costs are reduced.

Loss rate implied by the additional rider l_T^{rider}



Without background risk ($X_T^B = 0$) and a guarantee rate $g = 0.00$.



For lognormal background risk with $\mu_{X^B} = 0.05$ and $\sigma_{X^B} = 0.01$ and $X_0^B = 1$. ($g = 0.00$.)

Fig. 4.4 Loss rate l_T^{rider} for varying levels of risk aversion γ. The horizontal dashed line depicts the maximal loss rate $\bar{l}$. The vertical black (grey) dashed line refers to the risk aversion for which the borrowing constraints of the fair (feasible) contract are no longer binding. The time to maturity is $T = 10$. The other parameters are given as in Table 4.1.

In addition, observe that the loss rate is only increasing in the level of risk aversion γ after a critical value γ^*. The critical value is to be interpreted as the level of risk aversion γ^* where the borrowing constraints are neither binding for the investor of a feasible nor the investor of a fair contract, i.e. $\gamma^* = \max\{\gamma^{*,\text{fair}}, \gamma^{*,\text{feasible}}\}$. Since the investor of the feasible contract is more aggressive, the critical value is simply the critical level of risk aversion of the feasible investor, i.e. $\gamma^* = \gamma^{*,\text{feasible}}$. While the critical level is approximately $\gamma^* = 4$ without background risk, it is even $\gamma^* = 8.5$ in the case of a lognormal background risk. Due to the borrowing constraints, the loss rate of the additional rider may also decrease in the level of risk aversion γ, i.e. for $\gamma < \gamma^*$. The borrowing constraints, if binding, imply an utility loss, too. The loss is decreasing in the level of risk aversion γ, i.e. the impact of the borrowing constraints is the lower the higher the risk aversion. It is zero for risk aversions which are higher than the critical value. In particular, for risk aversion levels γ where $\gamma^{*,\text{fair}} < \gamma < \gamma^{*,\text{feasible}}$, the utility loss due to the borrowing constraints is decreasing in the case of a feasible contract but is constant (zero) in the case of a fair contract. In summary, the loss rate of the additional rider is increasing in γ for $\gamma > \gamma^* = \gamma^{*,\text{feasible}}$, it decreases in γ for $\gamma^{*,\text{fair}} < \gamma < \gamma^{*,\text{feasible}}$. For $\gamma < \gamma^{*,\text{fair}}$ the effect is ambiguous.

4.4.3 Realistic example

Recall that the critical levels of risk aversion are strongly influenced by the existence of background risk. Thus, it is important to assess if a realistic scenario implies that the loss rate of the additional rider is decreasing in the level of risk aversion or not. Based on realistic input data, we illustrate the utility loss for different types of investors. We consider three types of GMAB-investors which are defined by the proportion of retirement income stemming from the GMAB. These are called *high*, *medium* and *low* GMAB–investors. We still assume that the background value X_T^B is lognormally distributed, cf. Equation (4.25). Based on the scenario w ($w \in \{\text{high, medium, low}\}$), we define the initial amount $P^{(w)}$ invested in the GMAB. Along the lines of the above reasonings, the optimization is based on constant mix strategies. Here, the terminal wealth $TW^{(w)}$ can be represented as follows

$$\begin{aligned} TW_T^{(w)} = X_T^B + GMAB_T^{(w)} &= X_T^B + P^{(w)} \max\{V_T^{(1)}, e^{gT}\} \\ &= P^{(w)} \left(\frac{X_T^B}{P^{(w)}} + \max\{V_T^{(1)}, e^{gT}\} \right). \end{aligned}$$

$V_T^{(1)}$ denotes the payoff of a constant mix strategy with an initial investment of 1. For a CRRA investor, it is enough to consider[24]

$$\tilde{T}W_T^{(w)} = Z_T^{B,w} + \max\{V_T^{(1)}, e^{gT}\} \text{ where } Z_T^{B,W} = \frac{X_T^B}{P^{(w)}}.$$

A realistic parametrization for Z_T is motivated by

$$\begin{aligned} &\ln Z_T^B \sim N((\mu_{Z^B} - \frac{1}{2}\sigma_{Z^B}^2)T, \sigma_{Z^B}^2)T) \\ &\text{with } \mu_{Z_T^B}^w = \frac{1}{T} \ln \left(E\left[\frac{f^{\text{ret}}\, I_T\, a_T}{s(w) \sum_{i=0}^{T-1} e^{-rt_i} I_{t_i}} \right] \right) = \frac{1}{T} \left(\ln \frac{1}{s(w)} + \ln E\left[\frac{f^{\text{ret}}\, I_T\, a_T}{\sum_{i=0}^{T-1} e^{-rt_i} I_{t_i}} \right] \right). \end{aligned}$$

[24] Notice that the loss rate implied by the additional rider is still the same.

f^{ret} denotes the retirement factor, I_t denotes the income at time t of the accumulation phase with initial value I_0. We benchmark on the average income of the year 2010 in Germany, i.e. $I_0 = 32003$. Using the mean of the growth rate of the average income since 1952, the initial income is projected to time t_i. The standard deviation σ_{Z^B} is approximated by the standard deviation of the historical growth rate, i.e. it is set equal to 0.035. $s(w)$ denotes the savings factor of the investor with w ($w \in \{$high, medium, low). $a_T = \int_0^\infty e^{-ru} \, {}_u p_{65} \, du$ is the annuity factor of an individual aged 65 at the retirement time T. As usual, ${}_t p_x$ denotes the probability of a living aged x to survive the next t years. In particular, we vary the saving factor $s(w)$ to model our three types of investors.[25]

Income and Pension Parameters			
$\mu^w_{Z^B_T} - T \ln \frac{1}{s(w)}$	-0.0403971	T	20
σ_{Z^B}	0.035	f^{ret}	0.43
a_T	18.36	I_0	32.003

w	s(w)
low	7.2%
medium	8.5%
high	11.3%

Table 4.2 Income and Pension Parameters.

The relevant income and pension parameters are summarized in Table 6.1. The parameters are set according to German data on income, pension, saving rates as well as demographic data. The age of retirement is set to 65, i.e. the investor starts contributing in the GMAB with 45, i.e. we set $T = 20$. According to the German mortality tables from the Statistisches Bundesamt, the retirement factor (which is adjusted for longevity risk) is given by $f^{ret} = 0.43$, and the annuity factor is $a_T = 18.36$.

Table 6.3 shows the corresponding utility loss for the three types of investors and varying levels of risk aversion γ. Independent of the risk aversion the investor who relies most heavily on the GMAB at retirement, incurs the highest loss in utility. For non binding borrowing constraints it still holds: the higher the risk aversion the higher

[25] Notice that it is the proportion between investment in the GMAB and other sources of retirement income that drives the results. Therefore, the impact on the utility loss remains the same fixing the amount invested in the GMAB and varying the level of other income sources.

the loss in utility. For binding borrowing constraints the loss rate decreases, i.e. for reasonable risk averse investors the utility loss is lower due to the borrowing constraints. This implies that moderate risk averse investors which are the typical clientele of these products have to pay a price in terms of the utility loss for the additional flexibility. However, compared to lower risk averse investors or very risk averse investors the loss is smaller.

Loss rates for low, medium and high GMAB-investor investors

investor	$\gamma = 1.5$	$\gamma = 2.0$	$\gamma = 2.5$	$\gamma = 3.0$
high	37.1	36.7	36.0	35.3
medium	32.1	31.9	31.4	30.9
low	29.2	29.2	28.8	28.3
investor	$\gamma = 3.5$	$\gamma = 4.0$	$\gamma = 4.5$	$\gamma = 5.0$
high	34.5.7	33.7	32.9	32.3
medium	30.2	29.6	29.0	28.4
low	27.8	27.3	26.8	26.2
investor	$\gamma = 5.5$	$\gamma = 6.0$	$\gamma = 15$	$\gamma = 20$
high	31.5	30.8	31.6	33.0
medium	27.9	27.3	26.7	28.1
low	25.7	25.2	23.9	25.4

Table 4.3 The Table gives the loss rates of the low, medium and high GMAB-investor for varying risk aversion levels γ in basis points. The parameters are set as stated in Table 4.1 and 6.1.

4.5 Conclusion

This chapter has analyzed the investor's incentive to deviate from an optimal diversified investment strategy due to the insurance companies price setting. We consider products where the payoff is linked to the performance of an investment strategy and includes a minimum interest rate guarantee. In the case that the investor also receives the rider to decide on the investment strategy, the provider takes into account for the highest possible guarantee value. This is given by the strategy which maximizes the insurance put, i.e. the most risky one. The most risky strategy implies that the whole portfolio is invested in one asset only, i.e. it is a purely non-diversifying strategy. A risk averse investor who maximizes her expected utility faces two opposing effects. On one hand, she looses risk capital by paying an unfair price for the guarantee. On the other hand, deviating from the diversified strategy means taking more risk than optimal. In consequence, the investments become more aggressively which might be considered as an unwanted incentive effect. In addition, the investor herself faces a utility loss.

Surprisingly, it turns out that the loss rate which is implied by the contract rider is not monotonously increasing in the level of risk aversion. This is due to the borrowing constraints which are imposed on a private investor. While the loss rate is increasing in the case that the risk aversion is sufficiently high such that the borrowing constraints are not relevant for the expected utility maximization, this is not true as long as the borrowing constraints are binding. The critical level of risk aversion where the borrowing constraints are not binding any more also depends on the background risk of the investor. By means of realistic data, we quantify both, the increase in the risk structure and the utility losses which are caused by the additional contract rider.

Part II

Structured Investment Products

Chapter 5

Pricing and Upper Price Bounds of Relax Certificates

1

5.1 Introduction

Before the financial market crises the complexity of traded certificates increased significantly from year to year. One trend were products written on several instead of one underlying. Amongst them were the so–called relax certificates which can be interpreted as a generalized version of bonus certificates.[2] After the crises, the market volume of traded bonus certificates decreased from over 20% to 6%.[3] However, relax certificates are still traded, and an increasing number of certificates on several underlyings are issued under different names and product types. This chapter analyzes relax certificates as an example for complex certificates on several underlyings.

Normally, relax certificates[4] are written on three stocks belonging to a similar market segment, like blue chips or primary products. They are also traded on indices. Their payoff depends on whether and when any of the underlyings touches a lower barrier. As

[1] This chapter is based on a joint work of Nicole Branger, Antje Mahayni and Judith C. Schneider forthcoming in the Review of Managerial Science.

[2] Bonus certificates pay the maximum of the underlying value and a fixed payoff if the underlying never reaches a lower boundary until maturity. If the barrier is crossed, however, the investor instead receives the underlying.

[3] Cf. monthly reports of the Börse Stuttgart (EUWAX) and the monthly statistics of the Deutsche Derivate Verband (DDI).

[4] Similar products are also called Top-10-Anleihe, Easy Relax Express, Easy Relax Bonus, Multi-Capped Bonus or Aktienrelax. Furthermore, there are also relax certificates which bear some features of express certificates.

long as the barrier is not reached, the ”bonus payments” of the certificates correspond to those of a coupon bond where the coupon payments well exceed the current level of interest rates.[5] However, if the lower barrier is hit, all future payments from the bond component are cancelled. Instead, the investor receives the minimum of the prices of the underlyings at maturity. Relax certificates thus combine a knock-out component (the bond) and a knock-in component (the minimum claim).[6] For the time to maturity, a typical choice is three years and three month with reference dates every 13 months or a maturity of about one year with a single reference date at maturity.

Relax certificates are advertised as follows: The bonus payments are appealing even in sideways and slightly bearish markets. The risk of losing the bonus payments is low since this event is triggered by a significant loss in one of the underlying stocks. However, relax certificates are less attractive in highly bullish and highly bearish markets. In the first case, the investor would have been better off with a direct investment in the stocks. With relax certificates, she foregoes the participation in increasing stock prices.[7] In extremely bearish markets, the investor is also worse off. Here, she has to participate in the (highest) losses at the stock market, which certainly contradicts the naming ’relax’.

In the following chapter, we provide a detailed analysis of relax certificates. This chapter is related to the literature on the pricing of structured products, i.e. products that combine stocks or bonds with positions in derivatives. Burth et al. (2001) and Grünbichler and Wohlwend (2005) analyze the Swiss market and find that these products are overpriced both in the primary and in the secondary market. Wilkens et al. (2003) report similar findings for the German market. They find evidence for a so-called ’life-cycle hypothesis’: the overpricing is largest at and shortly after issuance of the products, when issuers mainly sell these products, and decreases over time, when

[5] Some examples for contracts which are currently traded will be given in Section 5.5.

[6] In the literature this minimum claim is also known as ”cheapest–to–deliver”, i.e. an option on the worse of n assets, see Wilkens et al. (2001).

[7] There are also certificates where the investor can participate in the development of the underlying assets if the terminal value of the worst performing stock is larger than the face value of the coupon bond.

issuers also start to buy back the products. Stoimenov and Wilkens (2005) furthermore find that the overpricing is the larger the more complex the product is. Muck (2006, 2007), Mahayni and Suchanecki (2006) and Wilkens and Stoimenov (2007) analyze the pricing of turbo certificates, i.e. barrier options, for the German market. Wallmeier and Diethelm (2008) and Lindauer and Seiz (2008) analyze (multi-) barrier reverse convertibles which are traded in Switzerland and are similar to the German relax certificates.

Recall that relax certificates can be interpreted as a knock-out coupon bond and a knock-in minimum claim. Closed-form solutions for standard barrier options are given by Rubinstein and Reiner (1991), Rich (1994) and Haug (1998). More exotic barrier options are, for example, considered in Kunitomo and Ikeda (1992) (two-sided barriers) and Heynen and Kat (1994) (outside barriers). For multi-asset barrier options, we refer to Wong and Kwok (2003) and Kwok et al. (1998). Closed-form solutions for pricing options on the minimum or maximum of two risky assets were first derived by Stulz (1982). An extension to more than two risky assets can be found in Johnson (1987).

The probability that at least one underlying reaches the barrier is important for the pricing and risk management of relax certificates. In the simple case of one underlying asset, the distribution of the first hitting time is well known in a Black-Scholes setup, cf. for example Merton (1973). It can be calculated using the reflection principle as shown in Karatzas and Shreve (1999) or Harrison (1985). For two underlyings, a semi–closed form solution is given in He et al. (1998) and Zhou (2001) where the distribution function is approximated by using an infinite Bessel function. We rely on these results in the following. The first hitting time distribution of more than two underlyings, however, cannot be given in closed–form for a general correlation structure.

Our main findings are as follows. The decomposition into a knock-out coupon bond and a knock-in minimum claim is useful to understand the structure of relax certificates. Usually, the contracts are designed such that relax certificates can be offered cheaper than the associated coupon bond. We call these relax certificates attractive and show that conditions for attractiveness can be summarized as follows: The bonus payments exceed a lower bound and/or the barrier level is below an upper bound. In this case, a trivial upper price bound is given by the corresponding coupon bond. This price bound can be tightened by subtracting the price of a put option on the minimum of the underlying assets with a strike price equal to the barrier.

In addition, we show that further price bounds can be determined by considering subsets of the underlyings. In the extreme case, we reduce the number of underlyings to one, so that the upper price bound can be calculated in closed–form in a Black-Scholes setup. This price bound is decreasing in the volatility of the underlying, and the lowest upper bound is thus given by using the stock with the highest volatility as underlying. Since the extreme case of one underlying obviously contradicts the basic idea of multiple underlyings, we also study higher dimensions. We show that tight but still tractable price bounds result from considering all subsets consisting of two underlyings.

In order to test the practical relevance of our theoretical results, we analyze relax certificates written on two or three underlyings which are currently traded at the market. For typical contract specifications, the price of the relax certificates is up to 10% lower than the price of the corresponding coupon bond. The risk that at least one of the underlying stocks hits the lower barrier can thus not be neglected and is highly economically significant. We also compare the market prices to the upper price bounds which are based on two underlyings only. It turns out that the market prices are well above these upper price bounds, which confirms that these contracts are overpriced and which also shows that the upper price bounds are rather tight.[8]

The remainder of the chapter is organized as follows. In Section 5.2, the payoff structure of relax certificates is defined and analyzed. In addition, we derive conditions on the contract parameters for which the certificates are attractive. This allows us to derive model independent upper price bounds in Section 5.3. In Section 5.4, we assume a Black–Scholes model and give the (exact) prices as well as (model-dependent) upper

[8] The price bounds are calculated in a Black-Scholes model. For attractive relax certificates, however, the price bounds would be even lower if one takes the possibility of (downward) jumps or default risk of the issuer into account.

price bounds. In particular, we give a tight upper price bound in semi–closed form and discuss the dependence of the prices and price bounds on the characteristics of the underlyings. A comparison to market prices can be found in Section 5.5. Section 5.6 concludes.

5.2 Product specification

5.2.1 Product specification

In general, a relax certificate is written on n underlying stocks, where n is equal to 2 or 3 for currently traded relax certificates. Let $S_t^{(j)}$ be the price of stock j at time t. For ease of exposition, we set the initial value of all stocks equal to one, i.e. $S_0^{(j)} = 1$ ($j = 1,\ldots,n$).[9] The continuously compounded risk-free rate is denoted by r. It is assumed to be constant and positive.

The payoff of the relax certificate depends on whether at least one of the stocks has hit its lower barrier m ($m < 1$), i.e. has lost the fraction $1-m$ of its value. Usually, m is chosen to be quite low, e.g. $m = 0.5$, so that this event constitutes a significant loss in this stock. The first hitting time of stock j ($j = 1,\ldots,n$) with respect to the barrier level m is denoted by $\tau_{m,j}$. The first hitting time of the portfolio of all underlying stocks is denoted $\tau_m^{(n)}$, i.e.

$$\tau_{m,j} := \inf\left\{t \geq 0, S_t^{(j)} \leq m\right\}, \quad \text{and} \quad \tau_m^{(n)} := \min\{\tau_{m,1},\ldots\tau_{m,n}\}. \tag{5.1}$$

If none of the underlyings reaches the level m, $\tau_m^{(n)}$ is set to $\tau_m^{(n)} = \infty$.

The relax certificate can be decomposed into two parts, a knock-out (RO) and a knock-in (RI) component. Its total payoff at maturity t_N is $\mathrm{RC}_{t_N}^{(n)} = \mathrm{RO}_{t_N}^{(n)} + \mathrm{RI}_{t_N}^{(n)}$, where we assume that payments before maturity are accumulated at the risk-free rate r. The set of all payment dates is denoted by $\underline{\underline{T}} = \{t_1,\ldots,t_N\}$, the current point in time is $t_0 = 0 < t_1$. If the barrier is not hit until $t_i \in \underline{\underline{T}}$ ($i = 1,...,N$), the investor receives a bonus

[9] This is in line with currently traded relax certificates where the minimum option is written on the *return* of the underlying stocks from time 0 to time t_N.

payment which is given by δ times the nominal value and which can be interpreted as a coupon payment. At maturity t_N, she also receives the nominal value of the certificate which we normalize to one. This part of the payoff can be interpreted as a knock-out component $\mathrm{RO}_{t_N}^{(n)}$

$$\mathrm{RO}_{t_N}^{(n)} = \sum_{i=1}^{N} \delta e^{r(t_N - t_i)} 1_{\{\tau_m^{(n)} > t_i\}} + 1_{\{\tau_m^{(n)} > t_N\}} \tag{5.2}$$

where 1 denotes the indicator function. If the barrier is hit before t_N, the investor forgoes all future bonus payments as well as the repayment of the nominal value. Instead, she gets the minimum of the n underlying stocks at the maturity date t_N. The payoff from this European knock-in component $\mathrm{RI}_{t_N}^{(n)}$ maturing at time t_N is given by

$$\mathrm{RI}_{t_N}^{(n)} = \min\left\{S_{t_N}^{(1)}, \ldots, S_{t_N}^{(n)}\right\} 1_{\{\tau_m^{(n)} \leq t_N\}}. \tag{5.3}$$

We summarize the payoff from the relax certificate in the following definition:

Definition 5.1 (Relax certificate). The compounded payoff of a relax certificate with nominal value 1, bonus payments δ, lower boundary m, payment dates $\underline{\underline{T}} = \{t_1, \ldots, t_N\}$, and n underlying stocks $S^{(1)}, \ldots, S^{(n)}$ is

$$RC_{t_N}^{(n)} = \sum_{i=1}^{N} \delta e^{r(t_N - t_i)} 1_{\{\tau_m^{(n)} > t_i\}} + 1_{\{\tau_m^{(n)} > t_N\}} + \min\left\{S_{t_N}^{(1)}, \ldots, S_{t_N}^{(n)}\right\} 1_{\{\tau_m^{(n)} \leq t_N\}}.$$

Note that we ignore any default risk of the issuer, which reduces the price of the certificate as compared to the prices without default risk. A detailed analysis of this issue is e.g. provided in Baule et al. (2008).

5.2.2 Attractive relax certificates

Relax certificates are advertised via rather high bonus payments and a price below the price of the corresponding coupon bond. We call these relax certificates *attractive*:

Definition 5.2 (Attractive relax certificate). A relax certificate is called attractive iff

$$\mathrm{RC}_0^{(n)} < \sum_{i=1}^{N} \delta e^{-rt_i} + e^{-rt_N}. \tag{5.4}$$

The *discount* as compared to the price of a coupon bond is achieved by the knock-out feature of the bond component. However, note that in case of a knock-out, the payoff is not replaced by zero but by the minimum of the stock prices at the maturity date. For the relax certificate to be attractive, the investor has to switch from a "higher" to a "lower" value in this case, i.e. the foregone future bond payments must be worth more than the minimum claim. A condition to ensure that this is indeed the case is given in the following lemma:

Lemma 5.1 (Attractive relax certificate: sufficient conditions). *A sufficient condition on the bonus payments δ and the lower barrier m to ensure that the relax certificate is attractive is given by*

$$m \leq \min_{\{j=0,\ldots,n-1\}} \delta \sum_{i:t_i>t_j} e^{-r(t_i-t_j)} + e^{-r(t_N-t_j)}. \tag{5.5}$$

In particular, a sufficient condition for Equation (5.5) to hold is given by

$$m \leq \frac{(1+\delta)e^{-rt_N}}{1+e^{-rt_N}}. \tag{5.6}$$

Proof. If the barrier is not hit, the payoff of the relax certificate is equal to that of a coupon bond. If the barrier is hit at time τ, the investor foregoes the future payments from this bond and receives a minimum claim instead. The terminal payoff of this claim is bounded from above by any of the stock prices, which implies that its value is also bounded from above by any of the stock prices. The smallest stock price at the hitting time τ is m, so that the value of the minimum claim at τ cannot exceed m.[10] Condition (5.5) ensures that this upper price bound on the minimum claim is smaller than the value of the coupon bond immediately after a coupon payment. In between the coupon dates, the price of the coupon bond increases and thus also exceeds m. Thus, the investor suffers a loss when the payments from the coupon bond are cancelled at τ

[10] In the case of gap risk due to jump or liquidity risk the lowest stock price can be lower than m.

and replaced by the minimum claim, so that the price of the relax certificate is indeed lower than the price of the coupon bond.

To prove the second part, note that

$$\min_{\{j=0,\ldots,n-1\}} \delta \sum_{i:t_i>t_j} e^{-r(t_i-t_j)} + e^{-r(t_N-t_j)} \geq \delta e^{-rt_N} + e^{-rt_N} \geq \frac{(1+\delta)e^{-rt_N}}{1+e^{-rt_N}}.$$

If Condition (5.6) holds, then m is smaller than the right hand side, which implies Condition (5.5). □

5.3 Risk-neutral pricing and upper price bounds

5.3.1 Risk-neutral pricing of relax certificates

In the following, we assume an arbitrage free market, i.e. the existence of a risk-neutral (pricing) measure $\mathbb{P}^*$. We do not restrict the type of model here. For the specific examples in Section 5.4, we will rely on a Black–Scholes–model.

For ease of exposition, we ignore any dividend payments of the stocks. Basically, dividends would reduce both the prices of attractive relax certificates and their price bounds. To get the intuition, note that dividends reduce the prices of the stocks and thus increase the probability that the lower barrier is hit, in which case the investor goes from a ”high” to a ”low” payoff for an attractive relax certificate. Since dividend payments also reduce the value of the minimum claim, the price of the relax certificate will decrease.

Let $\mathrm{RC}_{t_0}^{(n)}$ denote the price at t_0 of a relax certificate which is written on n underlying assets $S^{(1)},\ldots,S^{(n)}$. Pricing by no arbitrage immediately gives:

Proposition 5.1 (Price of a relax certificate). *The price at time t_0 ($t_0 = 0 < t_1$) of a relax certificate with bonus payments δ, lower barrier m, payment dates $\underline{\underline{T}} = \{t_1,\ldots,t_N\}$ and n underlying assets is given by $RC_{t_0}^{(n)} = RO_{t_0}^{(n)} + RI_{t_0}^{(n)}$. The prices of the components are*

$$RO_{t_0}^{(n)} = \delta \sum_{i=1}^{N} e^{-rt_i}\, \mathbb{P}^* \left(\tau_m^{(n)} > t_i\right) + e^{-rt_N}\, \mathbb{P}^* \left(\tau_m^{(n)} > t_N\right), \tag{5.7}$$

$$RI_{t_0}^{(n)} = E_{\mathbb{P}^*} \left[\int_{t_0}^{t_N} e^{-ru} C_u^{Min,n} dN_u\right] \tag{5.8}$$

where $N_t := 1_{\{\tau_m^{(n)} \le t\}}$ *and* $C_t^{Min,n} := \mathbb{E}_{\mathbb{P}^*}\left[e^{-r(t_N-t)} \min\left\{S_{t_N}^{(1)}, \dots, S_{t_N}^{(n)}\right\} \mid \mathscr{F}_t\right]$.

Proof. Pricing by no arbitrage immediately gives

$$\begin{aligned}\mathrm{RC}_{t_0}^{(n)} &= \delta \sum_{i=1}^{N} e^{-rt_i}\, \mathbb{P}^* \left(\tau_m^{(n)} > t_i\right) + e^{-rt_N}\, \mathbb{P}^* \left(\tau_m^{(n)} > t_N\right) \\ &\quad + \mathbb{E}_{\mathbb{P}^*}\left[e^{-rt_N} \min\left\{S_{t_N}^{(1)}, \dots, S_{t_N}^{(n)}\right\} 1_{\left\{\tau_m^{(n)} \le t_N\right\}}\right].\end{aligned}$$

Using the definition of N_t and iterated expectations yields

$$\begin{aligned}&\mathbb{E}_{\mathbb{P}^*}\left[e^{-rt_N} \min\left\{S_{t_N}^{(1)}, \dots, S_{t_N}^{(n)}\right\} N_{t_N}\right] \\ &= \mathbb{E}_{\mathbb{P}^*}\left[\int_{t_0}^{t_N} e^{-rt_N} \min\left\{S_{t_N}^{(1)}, \dots, S_{t_N}^{(n)}\right\} dN_u\right] \\ &= \mathbb{E}_{\mathbb{P}^*}\left[\int_{t_0}^{t_N} e^{-ru} \mathbb{E}_{\mathbb{P}^*}\left[e^{-r(t_N-u)} \min\left\{S_{t_N}^{(1)}, \dots, S_{t_N}^{(n)}\right\} |\mathscr{F}_u\right] dN_u\right].\end{aligned}$$

With the definition of $C_t^{\mathrm{Min},n}$, the pricing formula follows. □

The price of the knock-out bond component in Equation (5.7) depends on the distribution of the first hitting time $\tau_m^{(n)}$, i.e. the first time when one of the stocks hits the barrier. In the model of Black-Scholes, the first hitting time distribution for one stock is well known and can be derived using the reflection principle as in Karatzas and Shreve (1999) or Harrison (1985). For $n = 2$, Zhou (2001) derives a semi–closed form solution for the first hitting time by approximating the distribution function using an infinite Bessel function. For the special cases of uncorrelated stock prices or perfectly positively correlated stock prices, the distribution of the first hitting time for $n \ge 2$ follows from the one-dimensional case. In general, however, even semi-closed form solutions do not exist for $n \ge 3$, and the price of the knock-out component has to be calculated numerically.

The price (5.8) of the knock-in minimum claim depends on the joint distribution of the first hitting time and all stock prices at this first hitting time. Here, a closed form solution exists in the model of Black-Scholes and for $n = 1$. For $n \geq 2$, however, an analytical pricing formula no longer exists in general.

Even in the case of a simple Black–Scholes–type model setup, the prices of relax certificates thus have to be determined numerically in general. Possible methods are binomial or trinomial lattices – see e.g. Hull and White (1993) – or finite difference schemes – see e.g. Dewynne and Wilmott (1994) – which become rather time-consuming for more than one underlying. In this case, a Monte-Carlo simulation is usually preferred.[11]

5.3.2 *Upper price bounds based on coupon bonds*

The price of an attractive relax certificate is by definition lower than the price of the corresponding coupon bond. This gives the first model independent price bound and a trivial superhedge. This trivial superhedge can easily be tightened by selling a put option on the minimum of the stock prices with strike price m.

Proposition 5.2 (Semi-Static Superhedge). *Assume that δ and m satisfy Equation (5.6). Then, the following semi-static strategy is a superhedge for the relax certificate: At $t_0 = 0$, buy the corresponding coupon bond (with coupon payments δ and payment dates $\underline{\underline{T}}$) and sell a put option on the minimum of the stocks with maturity date t_N and strike m. If $\tau_m^{(n)} < t_N$, liquidate the portfolio at $\tau_m^{(n)}$ and use the proceeds to buy the cheapest underlying asset.*

Proof. Consider the case where one of the stocks hits the barrier first. At the hitting time $\tau_m^{(n)} < t_N$, the value of the hedge portfolio is $CB_{\tau_m^{(n)}} - P^{Min,n}_{\tau_m^{(n)}}$, where CB and $P^{Min,n}$ denote the value of the coupon bond and the price of the put option on the minimum of n stocks, respectively. The value of the bond is at least as large as the discounted value

[11] Notice that the barrier feature causes some problems for the Monte Carlo simulation, see Boyle et al. (1997).

of the payment at the maturity date, i.e. it is at least as large as $e^{-r(t_N-\tau_m^{(n)})}(1+\delta)$. The payoff of the put on the minimum stock price at t_N is

$$P_{t_N}^{Min,n} = \max\left\{m - \min\{S_{t_N}^{(1)},\dots,S_{t_N}^{(n)}\},0\right\}$$

and is bounded from above by m. At $\tau_m^{(n)}$, it thus holds that $P_{\tau_m^{(n)}}^{Min,n} \leq e^{-r(t_N-\tau_m^{(n)})}m$. This gives

$$\begin{aligned}CB_{\tau_m^{(n)}} - P_{\tau_m^{(n)}}^{Min} &\geq e^{-r(t_N-\tau_m^{(n)})}(1+\delta) - e^{-r(t_N-\tau_m^{(n)})}m \\ &= e^{-r(t_N-\tau_m^{(n)})}(1+\delta-m).\end{aligned}$$

With condition (5.6), it follows that

$$CB_{\tau_m^{(n)}} - P_{\tau_m^{(n)}}^{Min} \geq m.$$

Therefore, the value of the hedge portfolio at time $\tau_m^{(n)}$ is large enough to buy the cheapest asset, which is worth at most m at $\tau_m^{(n)}$. Obviously, this asset superhedges the minimum claim, which also holds true if it pays dividends.

In case none of the stocks hits the barrier, the put on the minimum expires worthless, and the bond component of the relax certificate is not knocked out. Thus, the payoffs of the hedge portfolio and of the relax certificate both coincide with the payoffs from the coupon bond and are thus equal. □

Corollary 5.1 (Upper bound on RC$_{t_0}^{(n)}$). *Assume that δ and m satisfy Equation (5.6). Then, an upper price bound for the relax certificate is given by*

$$RC_{t_0}^{(n)} \leq \sum_{i=1}^{N} \delta e^{-rt_i} + e^{-rt_N} - P_{t_0}^{Min,n}. \tag{5.9}$$

Proof. The proof follows immediately from Proposition 5.2. □

The semi–static superhedge in Proposition 5.2 can be simplified by considering only a subset of underlyings, as will be shown in Section 5.3.3. Looking at one underlying only leads to a semi-static hedge where only one plain-vanilla put option instead of the more exotic put on the minimum is needed. The optimal choice which gives the lowest initial capital is then the most expensive put.

An issuer who sells the relax certificate as a substitute for selling a coupon bond might follow yet another hedging strategy. As long as the barrier is not hit, he might just refrain from hedging at all. If the barrier is hit, however, he is no longer short a coupon bond but a minimum option. Then, he can hedge by taking a long position in the worst performing stock that is the stock which has first hit the barrier. This implies paying back the bond before maturity at a rather low level m.

5.3.3 Upper price bounds based on 'smaller' relax certificates

The next proposition shows that the price of an attractive relax certificate is decreasing in the number of underlyings. Considering a smaller number of underlyings thus gives an upper price bound.

Proposition 5.3 (Upper price bound: relax certificates on a subset of underlyings only). *Let $\mathscr{S} = \left(S^{(1)}, \dots, S^{(n)}\right)$ denote a set of underlyings. In addition, let $RC_{t_0}(\hat{\mathscr{S}})$ denote the price of a relax certificate with bonus payments δ, lower barrier m, payment dates $\underline{\underline{T}}$ and underlyings $\hat{\mathscr{S}}$ where $\hat{\mathscr{S}} \subseteq \mathscr{S}$. If condition (5.5) on the bonus payments δ and the barrier m holds, then*

$$RC_{t_0}(\mathscr{S}) \leq RC_{t_0}(\hat{\mathscr{S}}) \text{ for all } \hat{\mathscr{S}} \subseteq \mathscr{S}. \tag{5.10}$$

In particular, it holds

$$RC_{t_0}(\mathscr{S}) \leq \min_{k,l \in \{1,\dots,n\}} RC_{t_0}\left(S^{(k)}, S^{(l)}\right) \leq \min_{i \in \{1,\dots,n\}} RC_{t_0}\left(S^{(i)}\right). \tag{5.11}$$

Proof. Notice that the 'big' certificate on $\mathscr{S}$ is knocked out no later than the 'small' one on $\hat{\mathscr{S}}$. Depending on whether and when the two certificates are knocked out, there are three cases. First, if both certificates survive until maturity, their payments coincide. Second, if both are knocked out at the same point in time, the minimum claim resulting from the 'big' certificate is written on more underlyings and thus dominated by the minimum claim resulting from the 'small' certificate. Third, if the 'big' certificate is knocked out while the 'small' one still survives, the minimum claim resulting from the 'big' certificate is again dominated by the minimum claim on the smaller set of underlyings, which is by condition (5.5) dominated by the value of the attractive 'small' certificate. In all three cases, the value of the 'small' certificate is thus at least as high as the value of the 'big' certificate. This proves the first part of the proposition. The second part then follows as a special case. □

The above result is model-independent. Given the distribution of the first hitting time – for which there is a closed form solution in the model of Black-Scholes for $n = 1$ and a semi-closed form solution for $n = 2$ – the price of the knock-out component (5.7) can be calculated in closed form. For the price of the knock-in component (5.8) which depends on the joint distribution of the first-hitting time and the stock prices at the first hitting time, we now give an upper bound which depends on the distribution of the first hitting time only.

Proposition 5.4 (Upper price bound for knock-in part). *For $n \geq 2$, an upper price bound on the knock-in component is given by*

$$RI_{t_0}^{(n)} \leq m \int_{t_0}^{t_N} e^{-ru} f^{\mathbb{P}^*}_{\tau_m^{(n)}}(u)\, du. \tag{5.12}$$

where $f^{\mathbb{P}^*}_{\tau_m^{(n)}}$ denotes the density of the first hitting time $\tau_m^{(n)}$. In particular, this immediately implies that

$$RI_{t_0}^{(n)} \leq m\, \mathbb{P}^*(\tau_m^{(n)} \leq t_N). \tag{5.13}$$

Proof. Using the law of iterated expectations gives

$$\begin{aligned}
\mathrm{RI}_{t_0}^{(n)} &= \mathbb{E}_{\mathbb{P}^*}\left[e^{-rt_N} \min\left\{S_{t_N}^{(1)}, \ldots . S_{t_N}^{(n)}\right\} 1_{\{\tau_m^{(n)} \leq t_N\}}\right] \\
&= \mathbb{E}_{\mathbb{P}^*}\left[E_{P^*}\left[e^{-rt_N} \min\left\{S_{t_N}^{(1)}, \ldots . S_{t_N}^{(n)}\right\} \mid \mathscr{F}_{\tau_m^{(n)}}\right] 1_{\{\tau_m^{(n)} \leq t_N\}}\right] \\
&= \mathbb{E}_{\mathbb{P}^*}\left[E_{P^*}\left[\min\left\{\hat{S}_{t_N}^{(1)}, \ldots . \hat{S}_{t_N}^{(n)}\right\} \mid \mathscr{F}_{\tau_m^{(n)}}\right] 1_{\{\tau_m^{(n)} \leq t_N\}}\right]
\end{aligned}$$

where $\hat{S}_t := e^{-rt} S_t$. $\hat{S}$ is a $\mathbb{P}^*$-martingale, so that $\min\{\hat{S}^{(1)}, \ldots . \hat{S}^{(n)}\}$ is a $\mathbb{P}^*$-supermartingale. Together with the Optional Sampling Theorem it follows

$$\mathbb{E}_{\mathbb{P}^*}\left[\min\left\{\hat{S}^{(1)}_{t_N},\ldots.\hat{S}^{(n)}_{t_N}\right\} \mid \mathscr{F}_{\tau_m^{(n)}}\right] \leq \min\left\{\hat{S}^{(1)}_{\tau_m^{(n)}},\ldots.\hat{S}^{(n)}_{\tau_m^{(n)}}\right\} \leq m\, e^{-r\tau_m^{(n)}}.$$

This implies

$$\mathrm{RI}^{(n)}_{t_0} \leq m\int_{t_0}^{t_N} e^{-ru} f^{\mathbb{P}^*}_{\tau_m^{(n)}}(u)\,du.$$

The second bound then follows. □

As a consequence we can state the following theorem.

Theorem 5.1 (Upper price bound for $n \geq 2$**).** *For* $n \geq 2$*, an upper price bound on the relax certificate on the underlyings* $S = (S^{(1)},\ldots,S^{(n)})$ *is given by*

$$\min_{k,l\in\{1,..,n\}} \left\{ m\,\mathbb{P}^*\left(\min\{\tau_{m,k},\tau_{m,l}\} \leq t_N\right) + \delta\sum_{i=1}^{N} e^{-rt_i}\mathbb{P}^*\left(\min\{\tau_{m,k},\tau_{m,l}\} > t_i\right) \right.$$
$$\left. + e^{-rt_N}\mathbb{P}^*\left(\min\{\tau_{m,k},\tau_{m,l}\} > t_N\right) \right\}. \tag{5.14}$$

Proof. According to Proposition 5.3, it holds that

$$\begin{aligned}\mathrm{RC}^{(n)}_{t_0} &\leq \min_{k,l\in\{1,..,n\}} \left\{\mathrm{RC}_{t_0}(S^{(k)},S^{(l)})\right\} \\ &= \min_{k,l\in\{1,..,n\}} \left\{\mathrm{RO}_{t_0}(S^{(k)},S^{(l)}) + \mathrm{RI}_{t_0}(S^{(k)},S^{(l)})\right\}\end{aligned}$$

The value of the knock-out component follows from Proposition 5.1, while Proposition 5.4 gives an upper bound on the value of the knock-in minimum claim. Putting the results together gives Equation (5.14). □

The upper price bound for a relax certificate on n underlyings is based on the prices of all relax certificates on two underlyings only. We can also use relax certificates on three underlyings. However, it is not possible to determine the hitting time probabilities for $n \geq 3$ in (semi-)closed form. Therefore the tightest bounds for $n = 3$ which are not based on numerical approximations are achieved by using:

Lemma 5.2 (Semi closed–form bounds on survival probabilities for n=3). *The probability* $\mathbb{P}^*\left(\tau_m^{(3)} \leq t\right)$ *can be bounded from below and above as follows:*

$$\underline{\mathbb{P}^*}(\tau_m^{(3)} \leq t) \leq \mathbb{P}^*\left(\tau_m^{(3)} \leq t\right) \leq \overline{\mathbb{P}^*}(\tau_m^{(3)} \leq t)$$

where

$$\begin{aligned}\overline{\mathbb{P}^*}(\tau_m^{(3)} \leq t) = \min\big\{&\mathbb{P}^*(\min\big\{\tau_{m,1},\tau_{m,2}\big\} \leq t) + \mathbb{P}^*(\tau_{m,3} \leq t),\\ &\mathbb{P}^*(\min\big\{\tau_{m,1},\tau_{m,3}\big\} \leq t) + \mathbb{P}^*(\tau_{m,2} \leq t),\\ &\mathbb{P}^*(\min\big\{\tau_{m,2},\tau_{m,3}\big\} \leq t) + \mathbb{P}^*(\tau_{m,1} \leq t)\big\}\\ \underline{\mathbb{P}^*}(\tau_m^{(3)} \leq t) = \max\big\{&\mathbb{P}^*(\min\{\tau_{m,1},\tau_{m,2}\} \leq t),\\ &\mathbb{P}^*(\min\{\tau_{m,1},\tau_{m,3}\} \leq t), \mathbb{P}^*(\min\{\tau_{m,2},\tau_{m,3}\} \leq t)\big\}.\end{aligned}$$

Proof. It holds that

$$\mathbb{P}^*(\tau_m^{(3)} \leq t) = \mathbb{P}^*(\min\big\{\tau_{m,1},\tau_{m,2},\tau_{m,3}\big\} \leq t)$$

Notice that

$$\big\{\min\big\{\tau_{m,1},\tau_{m,2},\tau_{m,3}\big\} \leq t\big\} = \big\{\min\big\{\tau_{m,1},\tau_{m,2}\big\} \leq t\big\} \cup \big\{\tau_{m,3} \leq t\big\}$$

Using

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$$

immediately gives the upper bound. The lower bound follows from $P(A \cup B) \geq P(A)$. □

To derive an upper price bound on a relax certificate, $\mathbb{P}^*(\tau_m^{(3)} > t)$ is replaced by $(1 - \underline{\mathbb{P}^*}(\tau_m^{(3)} \leq t))$ while $\mathbb{P}^*(\tau_m^{(3)} \leq t)$ is replaced by $\overline{\mathbb{P}^*}(\tau_m^{(3)} \leq t)$. It is straightforward to show that the resulting upper price bound is higher than the one given in Theorem 5.1.

5.4 Numerical examples

5.4.1 Risk-neutral measure

For the specific examples, we rely on a Black–Scholes–type model setup with no dividends. Each stock price $S_t^{(j)}$ satisfies the stochastic differential equation

$$dS_t^{(j)} = \mu_j S_t^{(j)} dt + \sigma_j S_t^{(j)} dW_t^{(j)}, \tag{5.15}$$

where $W^{(j)}$ is a standard Brownian motion under the real world measure P. The Wiener processes are in general correlated, i.e. for $i \neq j$ it holds that $dW_t^{(i)}\, dW_t^{(j)} = \rho_{ij} dt$ where we assume constant correlations. Equation (5.15) implies that the dynamics of the stock prices under the risk neutral measure P^* are

$$dS_t^{(j)} = r S_t^{(j)} dt + \sigma_j S_t^{(j)} dW_t^{\mathbb{P}^*,(j)} \tag{5.16}$$

where $W^{\mathbb{P}^*,(j)}$ is a standard Brownian motion under $\mathbb{P}^*$.

5.4.2 Prices of relax certificates

For the model of Black-Scholes and in the case of one underlying, the first hitting time distribution is well known and was derived using the reflection principle as in Karatzas and Shreve (1999) or Harrison (1985). Using these formulas for the first hitting time, the price of the relax certificate can be calculated in closed–form:

Proposition 5.5 (Price of a relax certificate on one underlying). *For $n = 1$, the price $RC_{t_0}^{(1)}$ can be given in closed–form. The survival probability $\mathbb{P}^*(\tau_m^{(1)} \geq t)$ needed in Equation (5.7) to price the knock-out component is:*

$$\mathbb{P}^*(\tau_m^{(1)} \geq t) = N\left(\frac{-\ln\frac{m}{S_0} + \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) + e^{2\frac{r-\frac{1}{2}\sigma^2}{\sigma^2}\ln\frac{m}{S_0}} N\left(\frac{\ln\frac{m}{S_0} + \left(r - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right)$$

where N denotes the cumulative distribution function of the standard normal distribution. The minimum claim in Equation (5.8) reduces to the underlying itself, and the price of the knock-in component is

$$RI_{t_0}^{(1)} = m \int_{t_0}^{t_N} e^{-ru} f_{\tau_m^{(1)}}^{\mathbb{P}^*}(u)\, du \tag{5.17}$$

where the density $f_{\tau_m^{(1)}}^{\mathbb{P}^*}$ of the first hitting time $\tau_m^{(1)}$ is given in Proposition A.1 or Corollary A.1 respectively of Appendix A.2.1.

Proof. The expression for the density $f_{\tau_m^{(1)}}^{\mathbb{P}^*}$ is based on well known results which, for the sake of completeness, are given in Appendix A.2.1. For the knock-in component, first note that

$$C_t^{\mathrm{Min},1} = \mathbb{E}_{\mathbb{P}^*}\left[e^{-r(t_N-t)}\min\left\{S_{t_N}^{(1)}\right\} \mid \mathscr{F}_t\right] = \mathbb{E}_{\mathbb{P}^*}\left[e^{-r(t_N-t)}S_{t_N}^{(1)} \mid \mathscr{F}_t\right] = S_t^{(1)}.$$

In addition, we know that at the hitting time $\tau_m^{(1)} = u$ it holds that $S_u = m$. This gives

$$E\left[\int_{t_0}^{t_N} e^{-ru}C_u^{\mathrm{Min},1}dN_u\right] = mE\left[\int_{t_0}^{t_N} e^{-ru}dN_u\right] \; = \; m\int_{t_0}^{t_N} e^{-ru}f_{\tau_m^{(1)}}^{\mathbb{P}^*}(u)\,du.$$

□

For two or more underlyings, we rely on Theorem 5.1. The distribution of the first hitting time is known in semi-closed form for $n = 2$, where we rely on He et al. (1998) and Zhou (2001). This allows us to calculate the price of the knock-out component and an upper bound for the knock-in component. The resulting upper bound for the price of a relax certificate on two underlyings is also an upper price bound for relax certificates on more than two underlyings.

The distribution of the first hitting time is given in the next proposition.

Proposition 5.6 (Distribution of first hitting time for $n = 2$). *The distribution of the first hitting time* $\min\{\tau_{m,k}, \tau_{m,l}\}$ *is given by*

$$\begin{aligned}\mathbb{P}^*\left(\min\{\tau_{m,k}, \tau_{m,l}\} > t\right) = \frac{2}{\alpha t}e^{a_k \ln\left(\frac{S_0^{(k)}}{m}\right)+a_l \ln\left(\frac{S_0^{(l)}}{m}\right)+bt} \\ \sum_{n=1}^{\infty}\sin\left(\frac{n\pi\theta_0}{\alpha}\right)e^{-\frac{r_0^2}{2t}}\int_0^{\alpha}\sin\left(\frac{n\pi\theta}{\alpha}\right)g_n(\theta)d\theta.\end{aligned}$$

The parameters and the function g_n are defined in Corollary A.2 in Appendix A.2.2.

Proof. The survival probability $\mathbb{P}^*\left(\min\{\tau_{m,k}, \tau_{m,l}\} > t\right)$ follows from the results of He et al. (1998) and Zhou (2001). Details are given in Appendix A.2.2. □

The upper price bound in Theorem 5.1 results from looking at all subsets with two underlyings. If the relax certificate itself is written on two underlyings only, the knock-out component can be priced exactly, and only the knock-in part is approximated from above. Since for most realistic parameter values the “main part” of the product is explained by the knock-out part, the price bound is especially tight in this case which is illustrated by the following simulation study.

We illustrate our results in a Black-Scholes economy with short rate $r = 0.05$, volatilities $\sigma_1 = \sigma_2 = \sigma = 0.4$, and correlations between all assets set to $\rho = 0.25$. The prices are calculated using a Monte-Carlo simulation with 10.000 simulation runs and a step size of 100 steps per day.[12]

Our base contract is an attractive relax certificate written on two underlyings $S^{(1)}$ and $S^{(2)}$ with initial values $S_0^{(1)} = S_0^{(2)} = 1$. The time to maturity is 3 years, intermediate payment dates are $t_1 = 1$ and $t_2 = 2$ (years), the bonus payment is $\delta = 0.11$ and the barrier is $m = 0.5$. The probability that none of the stocks hits the barrier until time 3 is 71.11%. The price of the knock-out component is 0.8624, and the price of the knock-in minimum claim is 0.1276. The knock-out component thus represents a large part of the price of the relax certificate.

The upper bound for the knock-in minimum component follows from Proposition 5.4 and is equal to 0.1445. This gives an upper price bound of 1.0069 for the relax certificate, which exceeds the true price by less than 2%.

In a first step, we analyze the impact of volatility and correlation. Figure 5.1 gives the price, the survival probability, and the prices of the knock-out and the knock-in component as a function of volatility and for various correlations ρ. In line with intuition, the survival probability is decreasing in volatility and increasing in correlation. The same holds true for the price of the knock-out coupon bond, which is equal to the discounted sum of survival probabilities as can be seen in Equation (5.7). The price of the knock-in minimum component is more involved. A larger volatility or a lower correlation increases the probability that the barrier is hit and that the payoff from the knock-in component is positive. This leads to a larger price of the knock-in component in Equation (5.13). At the same time, a high volatility or a low correlation increases the probability for at least one very low terminal stock price, which reduces the price of

[12] To control the accuracy of the approximation, the simulation results for the survival probabilities and the prices of the knock–out component are compared to the exact closed-form solutions. These closed–form solutions, which are valid for one and two underlyings in the Black–Scholes model only, also allow for a quick calculation of the upper price bound.

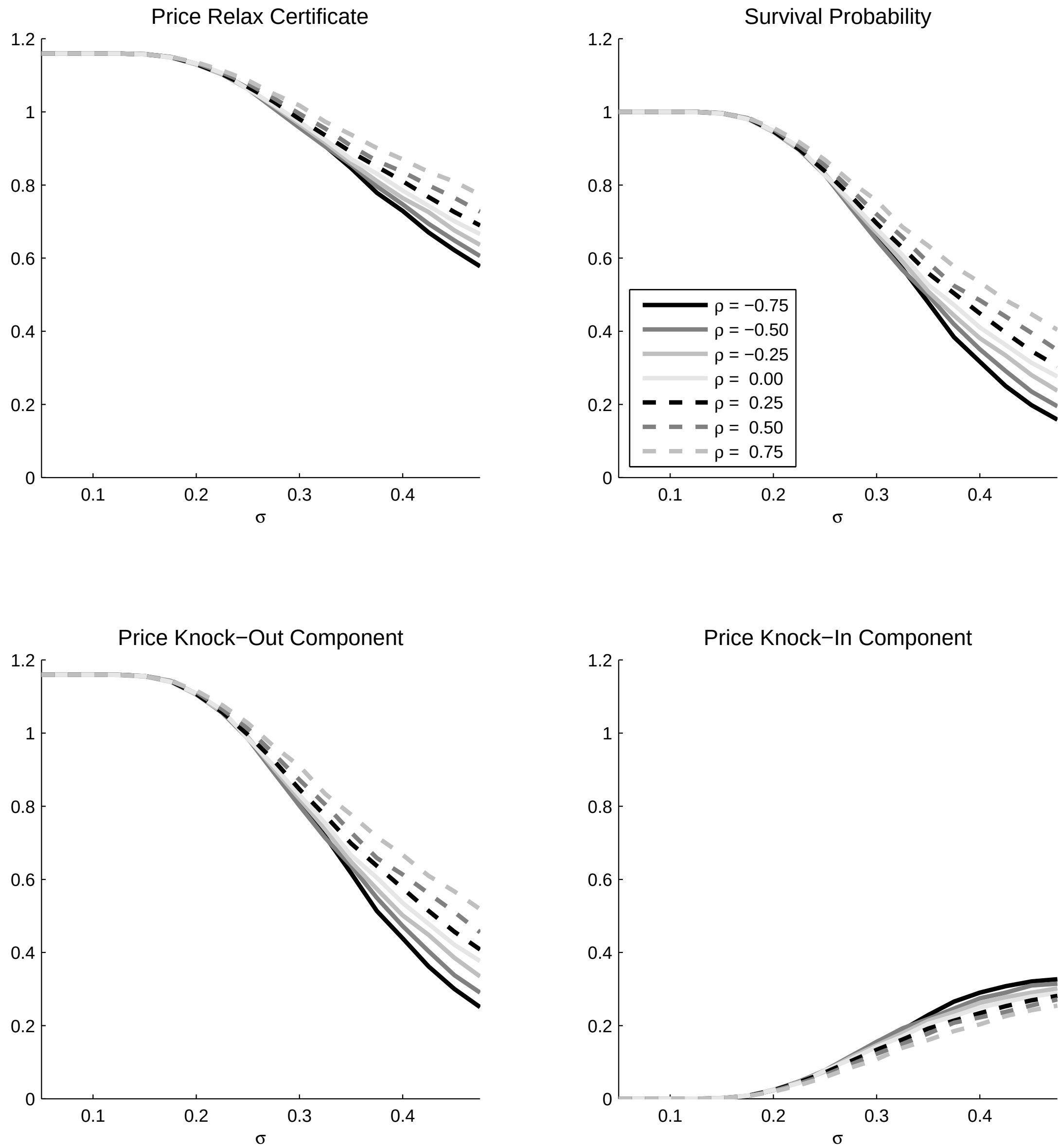


Fig. 5.1 Relax certificate on two underlyings: impact of volatility and correlation
The figure gives the price, the survival probability, the price of the knock-out component and the price of the knock-in component of a relax certificate as a function of the volatility of the two stocks for varying correlations. The parameters are $m = 0.5$, $\delta = 0.11$, $\underline{\underline{T}} = \{1,2,3\}$, $S_0^{(1)} = S_0^{(2)} = 1$ and $r = 0.05$.

the knock-in component. For our parameters, the first effect dominates, and the price of the knock-in component is increasing in volatility and decreasing in correlation.

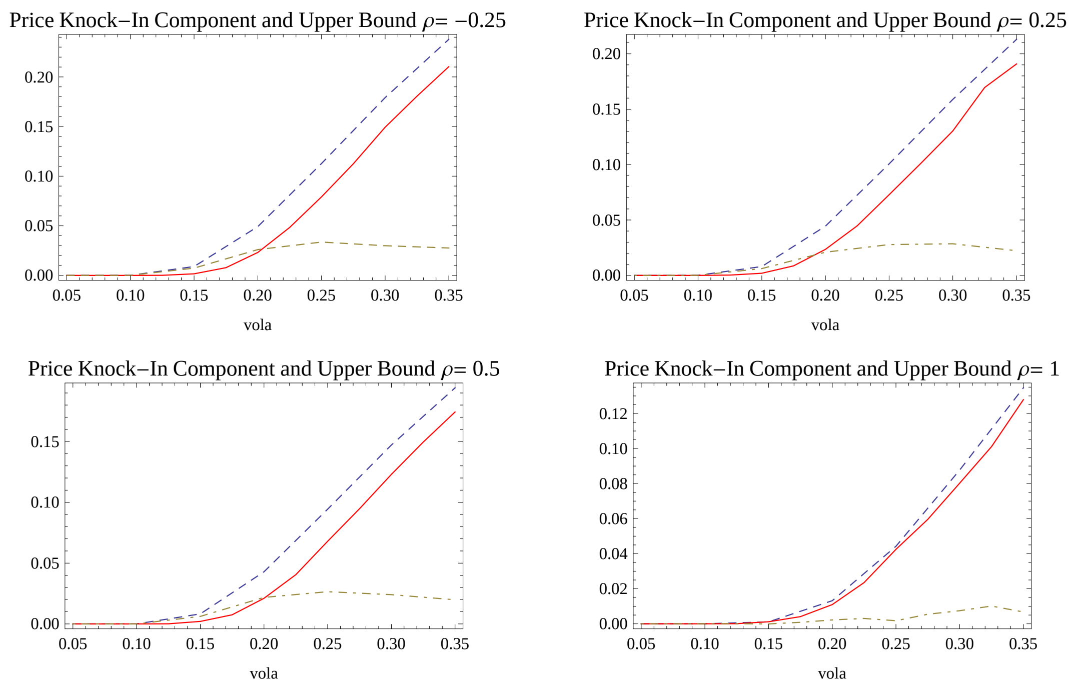


Fig. 5.2 Exact price vs. upper bound of knock-in part
The figure compares the exact price of the knock-in component (solid line) with the upper price bound derived in Proposition 5.4 (dashed line). The dash-dotted line shows the difference between the upper price bound and the exact price.

The results also show that the knock-out part represents a large part of the price of the relax certificate for nearly all correlations and volatilities. Its price contribution ranges from nearly 100% for a volatility of 0.1 and all correlations to at least 50% for all positive correlations and all volatilities. For negative correlations and very high volatilities ($\sigma \geq 0.4$) the knock-in part dominates because of the low survival probabilities. The difference between the upper price bound and the exact price of the knock–in part is illustrated in Figure 5.2. This difference is increasing in volatility and decreasing in correlation. The overestimation of the true price by the upper price bounds given in Proposition 5.4 and Theorem 5.1 respectively is rather small. It ranges from basically zero (for $\sigma \leq 0.1$ and all correlations) to 2.8% (for $\sigma = 0.35$ and $\rho = -0.25$) of the price of the relax certificate.

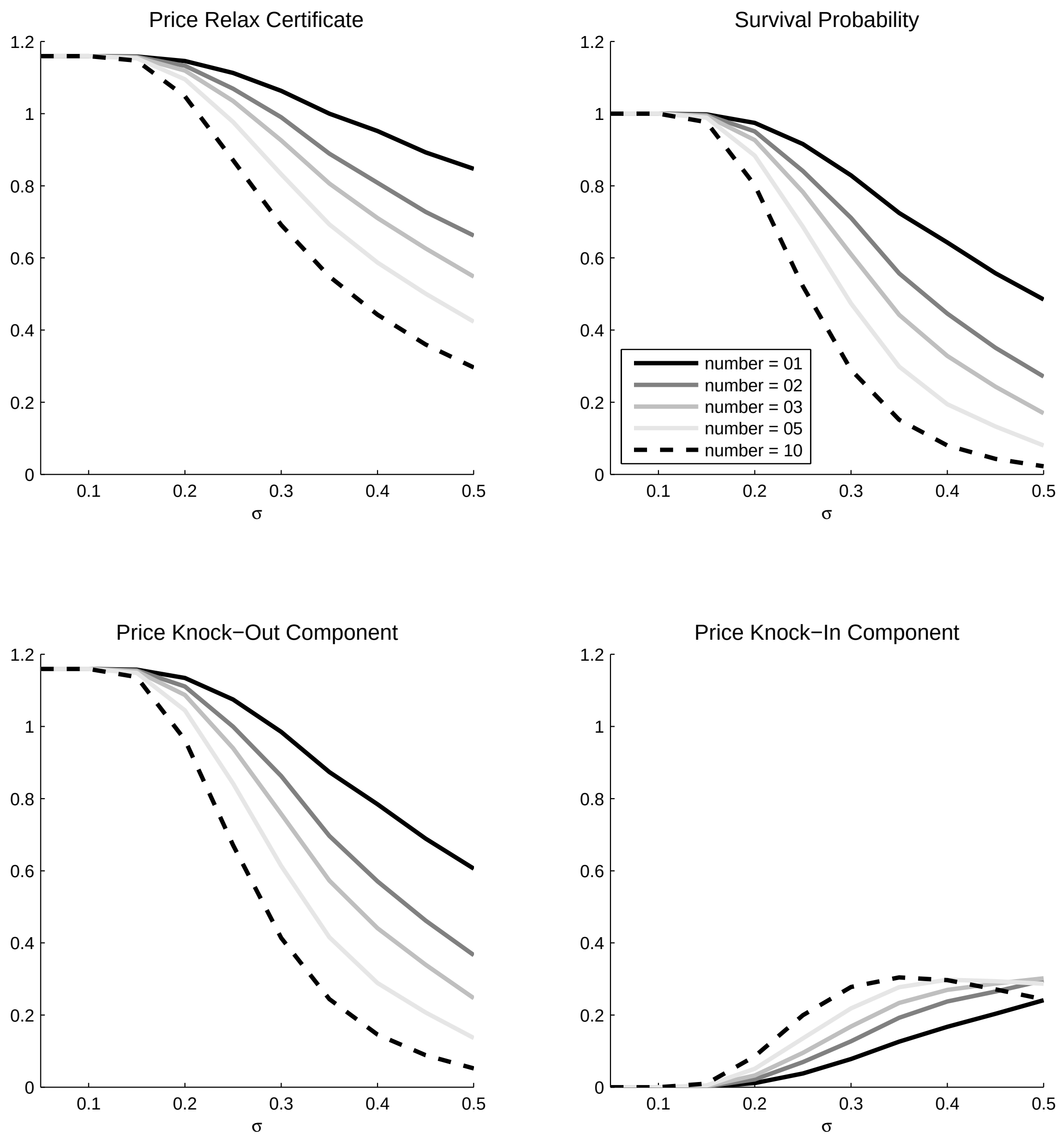


Fig. 5.3 Relax certificate on several underlyings: impact of volatility and number of underlyings
The figure gives the price, the survival probability, the price of the knock-out component and the price of the knock-in component of a relax certificate as a function of the volatility of the stocks for 1, 2, 3, 4, 5, and 10 uncerlyings. The parameters are $m = 0.5$, $\delta = 0.11$, $\underline{\underline{T}} = \{1,2,3\}$, $S_0^{(i)} = 1$, $\rho = 0.25$, and $r = 0.05$.

Secondly, we illustrate the impact of the number of underlyings, where we know from Proposition 5.3 that the price is decreasing in the number of underlyings. Figure 5.3 shows the survival probability and the prices of the relax certificate, the knock-out component and the knock-in component as a function of volatility for 1, 2, 3, 4, 5, and 10 stocks. In line with intuition, both the survival probability and the price of the knock-out component are the smaller the larger the number of underlyings. The price of the knock-in component is again more involved. While a larger number of underlyings increases the probability for a positive payoff from this component, it also increases the risk that at least one stock price at maturity is very low. For a small volatility, the first effect dominates, and the price of the knock-in component increases in the number of underlyings. For a large volatility, however, the price drops when the number of underlyings increases to ten. Furthermore, for ten underlyings the price of the knock-in component is no longer an increasing function of volatility, but decreases for large volatilities.

Figure 5.4 shows the impact of the time to maturity. In line with intuition, the survival probability is the lower the longer the time to maturity. For low volatilities, the decrease in the survival probability is rather small. For a bonus rate of 11% which well exceeds the risk-free rate of 5%, the price of the knock-out bond component increases in the time to maturity. In contrast, for high volatilities the decrease in the survival probability dominates, and the price of the knock-out component decreases in the time to maturity. Since the knock-out component accounts for a large part of the overall price of the relax certificate, we see the same dependence on time to maturity for the overall price. The price of the knock-in component again depends on the probability of being knocked in, which increases in the time to maturity, and on the payoff of the minimum claim, which decreases in the time to maturity. For the volatilities and times to maturity in our example, the first effect dominates, and the price of the knock-in component increases in the time to maturity.

Finally, we look at the impact of the barrier level m and the bonus payments δ, which is shown in Figure 5.5. The survival probability and the price of the knock-in component are both independent of the bonus payments δ. The survival probability is

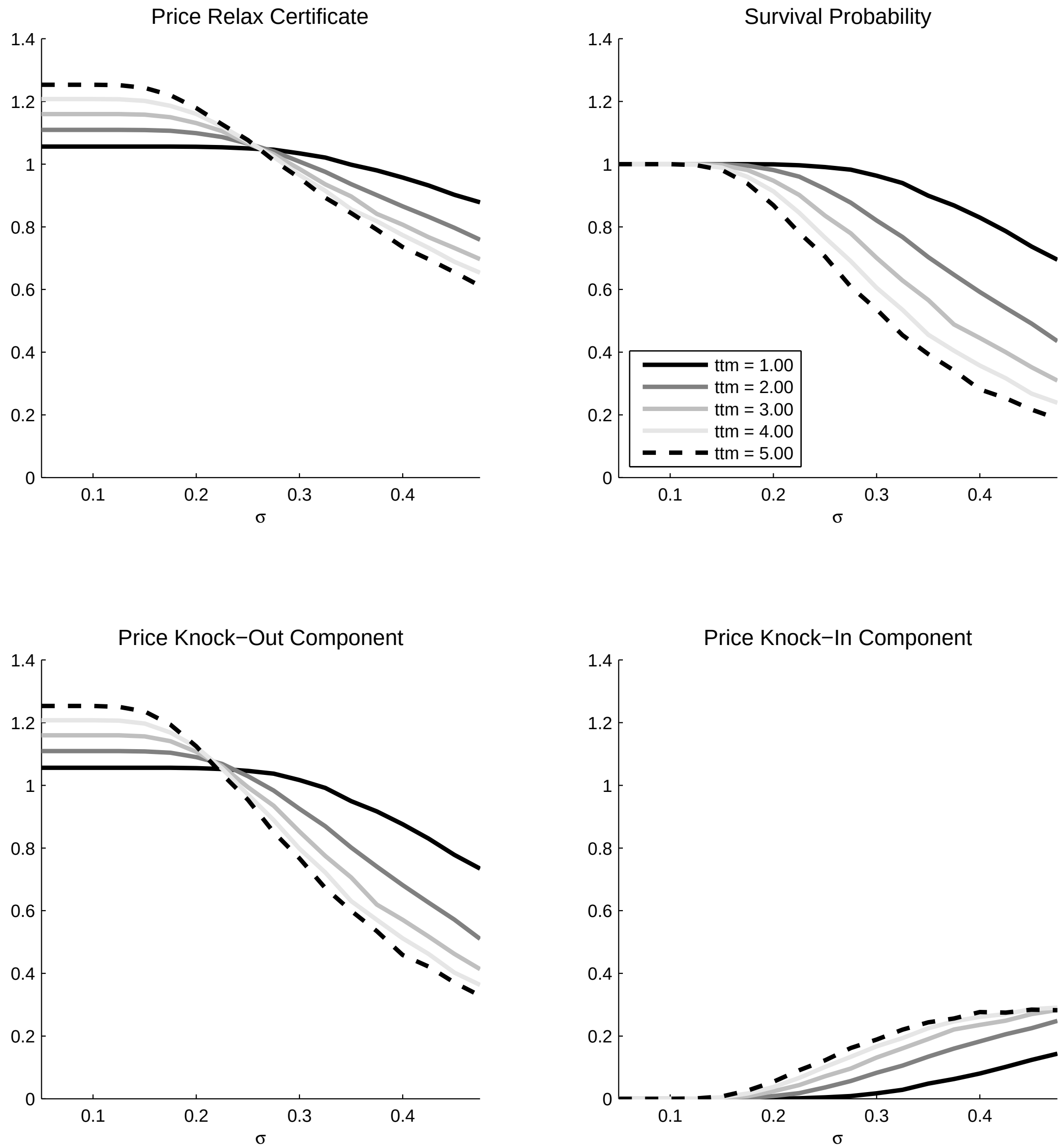


Fig. 5.4 Relax certificate on two underlyings: impact of volatility and time to maturity
The figure gives the price, the survival probability, the price of the knock-out component and the price of the knock-in component of a relax certificate as a function of the volatility of the two stocks for varying times to maturity. The parameters are $m = 0.5$, $\delta = 0.11$, $S_0^{(1)} = S_0^{(2)} = 1$, $\rho = 0.25$, and $r = 0.05$.

decreasing in m, while the price of the knock-in component increases in m. The price of the knock-out component, on the other hand, decreases in m. It furthermore increases in the level of the bonus payments.

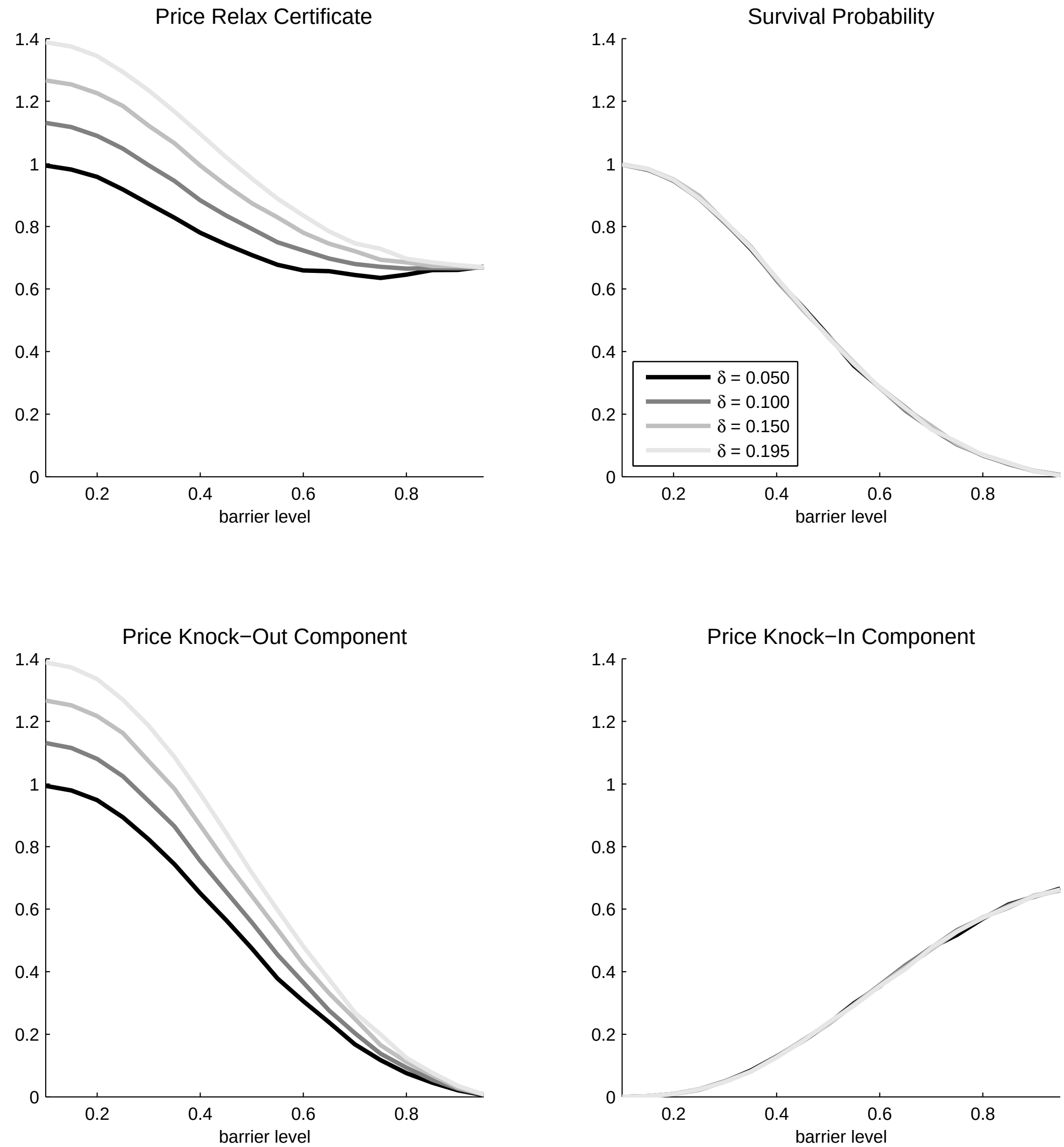


Fig. 5.5 Relax certificate on two underlyings: impact of barrier level and bonus payments
The figure gives the price, the survival probability, the price of the knock-out component and the price of the knock-in component of a relax certificate as a function of the barrier level m for varying bonus payments δ. The parameters are $\underline{\underline{T}} = \{1,2,3\}$, $S_0^{(1)} = S_0^{(2)} = 1$, $\sigma = 0.4$, $\rho = 0.25$, and $r = 0.05$.

5.5 Market comparison

5.5.1 Contract specifications

We now analyze some relax certificates issued in 2008 and 2009 and compare their issue prices to our price bounds. Table 5.1 gives the contract specifications of six typical certificates. All barriers are set to a rather low value (50% or 60%), so that at least one of the underlying stocks has to loose a high fraction of its initial value for the coupon bond to be replaced by the minimum claim. Furthermore, the bonus payments are large enough for all certificates to be attractive in the sense of Definition 5.2.

The table also gives the underlyings as well as the implied volatilities of at-the-money options on these underlyings with a time to maturity equal to the maturity of the certificate. The time to maturity of all contracts is below four years, and only one of the six certificates has intermediate payment dates.

Finally, the second column in Table 5.2 gives the issue price of the certificates. All certificates are issued one Euro above par.

5.5.2 Survival probabilities and price bounds

For an attractive relax certificate, the price of the corresponding coupon bond is a trivial upper price bound. The interest rates are inferred from the corresponding zero coupon bonds (swaps) via bootstrapping and are given in the last column of Table 1. The resulting prices of the coupon bonds are given in Table 5.2. For all certificates, the issue price is significantly lower than the price of the corresponding coupon bond. The risk that at least one of the stocks loses more than 50% respectively 40% of the initial value reduces the value of the relax certificate by 4% to 11%. This risk should thus not be neglected.

To assess the risk inherent in the relax certificate, we look at the (risk-neutral) probability that the barrier will not be hit. This survival probability is bounded from above by the survival probability for one or two underlyings. In the calculation, we set $\rho_{k,l} = 0.3$

	n	δ	m	underlyings	IV	payment dates	interest rate
C1 Commerz- bank	2	0.11	0.5	Daimler AG Siemens AG	0.33 0.35	14 months	0.0464
C2 HSBC	3	0.16	0.5	Daimler AG Siemens AG EON	0.33 0.35 0.27	17 months, 4 days	0.0464
C3 HVB	3	0.10	0.5	Allianz BASF Deutsche Post	0.32 0.25 0.30	13 months 26 months 39 months	0.0464 0.0458 0.0455
C4 HVB	3	0.11	0.5	EON Siemens AG Tui	0.23 0.30 0.60	39 months	0.0455
C5 Societé Géneral	3	0.35	0.5	S&P 500 DJ Euro STOXX NIKKEI 225	0.26 0.25 0.21	44 months, 4 days	0.0455
C6 WestLB	3	0.18	0.6	S&P 500 DJ Euro STOXX NIKKEI 225	0.26 0.25 0.21	18 months, 15 days	0.0464

Table 5.1 Summary of traded product specifications and interest rates.
For each certificate C1 - C6, the table gives the issuer, the number of underlyings n, the bonus payments δ and the lower barrier m. It also gives the underlyings and the implied volatilities of at-the-money options on these underlyings with a time to maturity equal to the time to maturity of the certificates. The last two columns contain the payment dates (only C3 has intermediate payments) and the risk-free rate for the corresponding investment horizons.

and $\sigma_k = \sigma_l = 0.3$ for C1–C4 and $\sigma_k = \sigma_l = 0.25$ for the remaining certificates.[13] The

[13] For all certificates, the implied volatilities of at least two underlying stocks as given in Table 5.1 are above 30%, so that a volatility of $\sigma = 0.3$ yields an upper bound for the survival probability. The same holds for the indices setting $\sigma = 0.25$

results show that adding a second underlying significantly increases the risk that the bond will be knocked out. They also confirm that the risk of a knock-out is rather high.

In the next step, we consider the upper price bounds that result from Theorem 5.1. They are also given in Table 5.2. For all certificates, the upper price bound of the knock-out coupon bond largely exceeds the upper bound on the value of the knock-in minimum claim. Furthermore, the resulting upper price bound is below the issue price for all but two certificates. If we account for dividend payments of the stocks, the upper price bound would even decrease further. The same holds true if we take credit risk into account.[14]

C	n	Issue price incl. load	corresp. coupon bond	Survival probability one under--lying	Survival probability two under-lyings	Upper price bound Knock-out comp.	Upper price bound Knock-in comp.	Upper price bound Price
C1	2	101.00	104.25	96.88%	93.60%	97.32	3.20	100.52
C2	3	101.00	108.68	94.97%	89.90%	96.11	5.25	101.36
C3	3	101.00	113.60	80.76%	65.35%	75.19	17.32	92.51
C4	3	101.00	116.32	81.77%	65.35%	80.03	17.32	97.35
C5	3	101.00	114.46	87.804%	70.62%	80.65	14.69	95.34
C6	3	1001.00	1101.66	91.57%	80.92%	891.10	114.00	1005.10

Table 5.2 Price bounds for traded certificates
For each certificate C1 - C6, the table gives the number of underlyings, the issue price, the price of the corresponding coupon bond, the survival probabilities based on one and two underlyings, and the upper price bounds based on two underlyings. The calculations are based on a volatility of $\sigma = 0.3$ for the stocks and $\sigma = 0.25$ for the indices. The correlation is $\rho = 0.3$.

For C1 and C2, we also calculate the upper price bounds using the implied volatilities of the underlyings and a correlation which ranges from -1 to 1. For $n = 1$, the upper price bound follows from Proposition 5.5, while we rely on Theorem 5.1 for

[14] In 2009, CDS spreads of Commerzbank e.g. increased to more than 100 basis points. In a very rough approximation, this would reduce the prices of our certificates by around 1%. We thank an anonymous referee for pointing out this example.

$n = 2$. For C1, we find that the issue price exceeds the lowest upper price bound for all correlation levels. For C2, the issue price is below the upper bound only if the correlation is larger than 0.85.

There are two possible conclusions. First, relax certificates may be overpriced in the market. This is in line with the empirical results of Wallmeier and Diethelm (2008) for the Swiss certificate market. Furthermore, the mispricing is the higher the higher the bonus payments (and thus the higher the discount due to the knock-out feature of the bond). We conjecture that the investors do not correctly estimate the risk associated with the barrier feature, but overweight the sure coupon.

Second, the model of Black-Scholes may not be the appropriate choice. If we include (on average downward) jumps as in Merton (1976), however, the knock-out probability increases. The resulting price bounds are lower than in the model of Black-Scholes such that the overpricing is even higher under a more realistic model setup. The same holds true if we account for default risk of the issuer, which again reduces the upper price bound calculated in a model. Dividend payments of the underlying, which we have not taken into account, have a similar effect and also reduce the upper price bound. Finally, our price bounds are based on two underlyings only, and they would be lower if we accounted for the larger number of underlyings. We thus conclude that it is hard to find a model-based motivation for the large prices of relax certificates at the market and that there is strong evidence that these contracts are indeed overpriced.

5.6 Conclusion

Relax certificates can be decomposed into a knock-out coupon bond and a knock-in minimum claim on the underlying stocks. The contracts are designed such that relax certificates can be offered at a discount compared to the associated coupon bond. Formally, this gives a condition on admissible (or *attractive*) contract parameters in terms of the barrier and the bonus payments.

The knock-out/knock-in event takes place when the worst-performing of the underlying stocks hits a lower barrier. Nevertheless, our analysis shows that the probability of a knock-out cannot be neglected and induces a significant price discount of the relax certificate as compared to the corresponding coupon bond. The risk is the larger the higher the volatility of the underlyings, the lower their correlation, the larger the number of stocks the certificate is written on, and the longer the time to maturity.

In general, numerical methods are needed to price relax certificates, and even in the Black-Scholes model closed form solutions only exist for one underlying. However, closed-form or semi-closed form solutions are available for upper price bounds. A trivial upper price bound is given by the corresponding coupon bond. Furthermore, the price of a relax certificate on several underlyings is bounded from above by the price of the (cheapest) relax certificate on a subset of underlyings. We show that two underlyings allow to achieve meaningful and tractable price bounds. The most likely candidates to give this lowest upper price bound are the relax certificates on the most risky assets and/or the assets with the lowest correlation between the underlyings.

Finally, we test the practical relevance of our theoretical results by comparing the price bounds to market prices. The upper price bounds are calculated based on the implied volatilities of call options on the respective underlyings. It turns out that relax certificates which are currently traded are significantly overpriced. This result is true for nearly all correlation scenarios.

Chapter 6

Sub-optimal investment strategies and mispriced put options under borrowing constraints

6.1 Introduction

This chapter considers investment products which guarantee some minimum level of wealth (downside protection) while, at the same time, participating in the potential profit of an underlying investment strategy. Such strategies are well known as portfolio insurance strategies. Two prominent examples are the constant proportion portfolio insurance (CPPI) and the protective put, i.e. the option-based portfolio insurance (OBPI). The concept of option–based portfolio insurance is already introduced in Leland and Rubinstein (1976) and Brennan and Schwartz (1976) .[1] The constant proportion portfolio insurance is introduced in Black and Jones (1987).[2] In practice, the CPPI strategy is the preferred choice, especially for Riester Products, e.g. DWS Riester Rente, or for Garantiefonds. The protective put can be found in certificates, e.g. DAXplus Protective Put.

In the context of a diffusion model framework and a complete market El Karoui et al. (2005) show that given an exogenous guarantee the optimal payoff is the unrestricted optimal investment strategy backed-up by a put option. The price of the put option corresponds to the reduction in the initial amount invested in the optimal investment strategy, this defines a *fair* contract. The result holds independently of the utility func-

[1] In the original article of Leland and Rubinstein (1976) the put options were replicated according to the Black-Scholes formula.

[2] For the basic procedure of the CPPI see also Merton (1971). For a detailed overview on the related literature we refer to Balder and Mahayni (2010).

tion. In the special case of a HARA utility function, the solution reduces to a strategy which justifies the CPPI approach. An analysis of the CPPI and the OBPI by means of utility theory is conducted by Balder and Mahayni (2010). Their approach is most similar to ours.

The investment strategy underlying the put option can be any dynamic self-financing strategy, e.g. in the Black-Scholes model and for a CRRA utility the optimal strategy is a constant mix. This implies a wider choice of product designs for option based portfolio insurance. Our paper extends the previous literature in two aspects. We place our analysis in a complete financial market model where the price dynamics are given by a diffusion process with constant parameters, i.e. a geometric Brownian motion (GBM). For ease of the exposition we let the retail investor pay the entire costs for the contract at inception.[3] We assume that the investor is restricted by borrowing and short selling restrictions. However, it is reasonable to assume that the issuer of these products can borrow money. Therefore, we do only restrict the investment fractions of the optimal retail investor's strategy. For the CPPI this implies that the portfolio weights are capped, thus the strategy becomes path-dependent. In contrast, for the protective put strategy, the borrowing constraints do restrict the portfolio weights of the investment strategy underlying the put option. The optimal unrestricted payoff remains a constant mix strategy, but where the riskless rate is increased for calculatory purposes so that the borrowing constraints are met.[4] In particular, the investment fractions do not change if one is forced to buy the guarantee by paying a fraction of the initial investment. However, there might arise another problem deducting the price for the guarantee from the initial premium. We argue that due to market conditions, mispriced put options, implied volatility or worst case pricing, the contract does not have to be *fair*, i.e. the price for the guarantee does not need to coincide with the endogenously determined price for the guarantee implied by the optimal payoff. This is similar to the reasoning in the paper by Zagst and Kraus (2009). They argue that whereas the CPPI represents

[3] This is quite common for certificates whereas Riester contracts usually come along with periodic premiums.

[4] See e.g.Tepla (2001) for a detailed analysis of borrowing and short selling restrictions.

a dynamic investment strategy such that historical volatility is the right choice, the protective put is static in the sense that the put options are bought at inception of the contract. In fact, implied volatility usually exceeds historical volatility, thus the put options are more expensive compared to the Black-Scholes price based on the historical volatility or the corresponding hedging strategy in the underlying market. Thus, the deduction from the initial investment of the investor differs from the *fair* deduction which would be implied by the underlying strategy. To compare the dynamic CPPI and the protective put strategy, we rely on expected utility theory where the guarantee is given exogenously. In particular, we compare the different payoffs by means of the loss rate for a CRRA investor who gains utility from terminal wealth only. Already the introduction of borrowing constraints is enough to reduce the loss in utility due to a CPPI and capped CPPI such that it can be almost neglected. Using mispriced puts the loss in utility is quite substantial. The subsequent sections are structured as follows: Section 6.2 summarizes some general results in utility theory and defines the payoffs to be analyzed. Based on this results Section 6.3 compares the optimal CPPI and the optimal protective put by means of utility losses. Section 6.4 concludes.

6.2 Optimal portfolio selection with terminal wealth guarantee and borrowing constraints

We consider an economy with n risky assets $S_1,...,S_n$ which are driven by continuous diffusion processes and one risk free asset $S_0 = e^{-rT}$ $(r > 0)$ defined on a probability space $(\Omega, \mathscr{F}, (\mathscr{F})_{t\in[0,T]}, \mathbb{P})$ where trading terminates at time T. In particular, we assume that the price processes of the risky assets are given by correlated geometric Brownian motions, i.e

$$dS_{t,i} = S_{t,i}\left(\mu_i\,dt + \sum_{j=1}^{N} b_{ij}\,dW_{t,j}\right), \quad S_{0,i} = s_i \tag{6.1}$$

where $W = (W_{t,1}, \ldots, W_{t,N})_{0\le t\le T}$ denotes a standard Brownian motion with respect to the *real world* measure $\mathbb{P}$. μ_i and b_{ij} are constants and we assume that $\mu_i > r \ge 0$. In particular, we use the notation $\sigma_i^2 := \sum_{j=1}^N b_{ij}^2$ and $\sigma_{ij} := \sum_{k=1}^N b_{ik}b_{jk}$. Thus, the market is complete i.e. there exists a uniquely defined martingale measure $\mathbb{P}^*$ such that W^* is a $\mathbb{P}^*$–Brownian motion and

$$dS_{t,i} = S_{t,i}\left(r\,dt + \sum_{j=1}^N b_{ij}\,dW_{t,j}^*\right), \;\; S_{0,i} = s_i. \tag{6.2}$$

Every trading strategy in the $n+1$ assets can be described by a predictable process $\phi_t = (\phi_{0,t}, ..., \phi_{n,t})$. In particular, we restrict ourselves to self–financing strategies, i.e. the discounted value process of a self–financing strategy V_t must be a martingale under $\mathbb{P}^*$. This implies that the investor can choose the portfolio weights $\pi_t = (\pi_{1,t}, ..., \pi_{n,t})$ to invest in the n risky securities at different points in time on the interval $[0,T]$ where the fraction invested in the risk free asset is defined by $(\mathbf{1}-\pi_t)'\mathbf{1} = \pi_{0,t}$. As argued above, we consider products which provide the investor participation in an investment strategy plus a certain guaranteed amount at maturity. Hence, along the lines of El Karoui et al. (2005) the optimization problem given a utility function $u(w)$ where $u'(w) > 0$ and $u''(w) < 0$ is defined by

$$\max \mathbb{E}_{\mathbb{P}}\left[u(V_T)\right] \;\; \text{under the constraints } V_T > G, \text{and } \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}V_T\right] = V_0, \tag{6.3}$$

over all self-financing portfolios.

Although, the optimal investment strategy for a retail investor depends on his preferences and the model setup, the terminal wealth guarantee always leads to a protective put. In particular,

Lemma 6.1 (El Karoui et al. (2005)). *Under a terminal wealth guarantee G, the optimal solution is given by the optimal portfolio of the unconstrained solution V_T^* backed up by a protective put such that the budget constraint is met, i.e. the optimal payoff is*

$$WG_T^*(G,\alpha^*) = \alpha^* \frac{V_T^*}{V_0} + [G - \alpha^* \frac{V_T^*}{V_0}]^+ \tag{6.4}$$

$$\textit{s.t. } \alpha^* \textit{ satisfies } e^{-rT}\mathbb{E}_{\mathbb{P}^*}\left[\alpha^* \frac{V_T^*}{V_0} + [G - \alpha^* \frac{V_T^*}{V_0}]^+\right] = V_0. \tag{6.5}$$

This holds in every arbitrage-free and complete market where asset prices are driven by continuous diffusion processes independent of the specific utility function.

Proof. See El Karoui et al. (2005) and for the representation in terms of an investment fraction α Branger et al. (2010). □

In the following, the guaranteed part G is defined by a guaranteed rate g with $g < r$ prevailing on the initial investment V_0, i.e.

$$G := V_0 e^{gT}. \tag{6.6}$$

Practical meaningful strategies can be derived for a CRRA investor and a HARA investor in a Black-Scholes setup.

6.2.1 CRRA Utility

The CRRA utility function allows that the optimization problem of the investor can be expressed in terms of maximizing the expected utility of the wealth increment. The optimization problem does not change if the initial wealth is changed in a multiplicative way, i.e.

$$u^{\text{CRRA}}(cw) = \frac{(cw)^{1-\gamma}}{1-\gamma} = c^{1-\gamma} u^{\text{CRRA}}(w).$$

Together, a Black-Scholes model and CRRA utility imply that the optimal investment strategy is a constant mix strategy.

Lemma 6.2 (Constant mix). *Let $\varphi \in \Phi^{CM}$ where Φ^{CM} denotes the class of constant mix (CM)–strategies where the portfolio weights $\pi_i := \frac{\varphi_{t,i} S_{t,i}}{V_t(\varphi)}$ are constant for all $t \in [0,T]$. Then, $V(\varphi)$ ($V(\pi)$, respectively) is lognormal with*

$$\mu_{V_{CM}} = r + \sum_{i=1}^{n} \pi_i(\mu_i - r) \text{ and } \sigma^2_{V_{CM}} = \sum_{i=1}^{n} \sum_{j=1}^{N} \pi_i \pi_j \sigma_{i,j}. \tag{6.7}$$

Proof. V^{CM} is a lognormal asset, i.e.

$$dV_t(\pi) = V_t(\pi)\left(\left(1-\sum_{i=1}^{n}\pi_i\right)r\,dt + \sum_{j=1}^{n}\pi_i\frac{d\,S_{t,i}}{S_{t,i}}\right)$$
$$= V_t(\pi)\left(\left(1-\sum_{i=1}^{n}\pi_i\right)r\,dt + \sum_{i=1}^{n}\pi_i\left(\mu_i\,dt + \sum_{j=1}^{n}b_{ij}\,dW_{t,j}\right)\right)$$

where $\sum_{i=1}^{n}\pi_i\sum_{j=1}^{n}b_{ij} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\pi_i\pi_j\sigma_{ij}}$ such that the solution to the stochastic differential equation by reducing the dimensionality to a one–dimensional standard $\mathbb{P}$–Brownian motion $\tilde{W}$ is

$$V_T^{CM} = V_0^{CM} e^{\left(\mu_{V_{CM}} - \frac{1}{2}\sigma^2_{V_{CM}}\right)T + \sigma_{V_{CM}}\tilde{W}_T}.$$

As argued above, in a Black-Scholes model the asset price increments are assumed to be independent and identical distributed. In particular, reducing the initial wealth in order to buy a guarantee does not change the optimization problem of a CRRA investor given that the guarantee is priced according to Equation (6.5), i.e. is priced fairly. The expected utility with a terminal wealth guarantee can be stated in closed form.

Proposition 6.1 (Expected Utility Terminal Guarantee). *Under a terminal wealth guarantee G and if V is a lognormal process with constant mean μ_{VCM} and constant diffusion coefficient σ_{VCM}, then the expected utility of a CRRA–investor is given by*

$$\mathbb{E}_{\mathbb{P}}\left[u(WG_T(G,\alpha))\right] = \mathbb{E}_{\mathbb{P}}\left[u(\alpha\frac{V_T^{CM}}{V_0} + [G-\alpha\frac{V_T^{CM}}{V_0}]^+)\right] \tag{6.8}$$
$$= \frac{1}{1-\gamma}\left[(\alpha V_0)^{1-\gamma}e^{(1-\gamma)(\mu_{VCM}-\frac{1}{2}\gamma\sigma^2_{VCM})T}\right.$$
$$\mathcal{N}\left(h^{(1)}\left(0,\frac{\alpha V_0}{Ge^{(-\mu_{VCM}+\gamma\sigma^2_{VCM})T}}\right)\right)$$
$$\left.+\,G^{(1-\gamma)}\mathcal{N}\left(-h^{(2)}\left(0,\frac{\alpha}{Ge^{(-\mu_{VCM})T}}\right)\right)\right] \tag{6.9}$$

$$\textit{where } h^{(1)}(t,z) := \frac{\ln z + \frac{1}{2}\sigma_{VCM}t}{\sigma_{VCM}\sqrt{t}} \textit{ and } h^{(2)}(t,z) := h^{(1)}(t,z) - \sigma_{VCM}\sqrt{t}. \tag{6.10}$$

Proof. A proof of this proposition is given in Appendix A.3.1.

The contract is called *fair* iff

Definition 6.1 (Fair investment fraction). W.l.o.g., let $V_0 = 1$. If the process V is lognormal, then a fair contract is given by the condition that at inception of the contract $t = 0$

$$\alpha^*(\sigma_{VCM}) = \frac{1 - Ge^{-rT}\mathcal{N}\left(-\frac{\ln\frac{\alpha^*}{G}+rT-\frac{1}{2}\sigma_{V_{CM}}^2 T}{\sigma_{V_{CM}}\sqrt{T}}\right)}{\mathcal{N}\left(\frac{\ln\frac{\alpha^*}{G}+rT+\frac{1}{2}\sigma_{V_{CM}}^2\sqrt{T}}{\sigma_{V_{CM}}\sqrt{T}}\right)}. \tag{6.11}$$

Notice that, in general $\alpha(\sigma)$ decreases if G increases or if σ increases. Thus, a higher volatility $\tilde{\sigma}$ than the volatility of the optimal investment strategy σ^* results in $\tilde{\alpha}(\tilde{\sigma}) \leq \alpha^*(\sigma^*)$. In general, if the volatility for pricing the put option deviates from the volatility implied by the conducted optimal underlying investment strategy of the investor, the contract is no longer *fair*. Thus, expected utility is no longer maximized. In particular, a lower $\tilde{\alpha}(\tilde{\sigma})$ than the *fair* $\alpha^*(\sigma^*)$ implies a loss in risk capital for the investor.

As argued above the retail investor faces borrowing constraints. Without terminal wealth guarantee, the optimal solution is not impeded by the introduction of borrowing constraints, i.e. the optimal strategy is still a constant mix strategy where the risk-free rate is *artificially* increased depending on the risk aversion of the retail investor. [5] As we only restrict the investment strategy for the retail investor, this also holds true for the protective put and the optimal solution still implies a constant mix strategy. In particular, the portfolio weights are defined by:

$$(\pi^{*,BC})'\mathbf{1} \leq 1.$$

For constant mix strategies the only risk parameters are the portfolio drift μ_{VCM} and the portfolio volatility σ_{VCM}. Together with CRRA utility, the investors' optimal un-

[5] In fact, Short–selling constraints imply a two step procedure where a subset of attainable risky assets is determined in the first step and then the procedure is to continue as if unrestricted. Therefore, we assume that the first step is already performed. A solution which can be adapted to our setting is provided in Tepla (2001) where the technical aspects are pointed out. However, Tepla (2001) considers the market to be incomplete.

constrained portfolios are instantaneously mean-variance efficient.[6] Thus, the problem reduces to a static optimization problem, i.e. it is enough to consider all tuples $(\mu_{VCM}, \sigma_{VCM})$ which lead to the highest μ_{VCM} for a given volatility σ_{VCM}:

Proposition 6.2. *The set of efficient $(\mu_{VCM}, \sigma_{VCM})$ for an agent with risk aversion γ and utility function $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$ is*

(i)in case of non-binding borrowing constraints

$$\mu_{V_{CM}} = r + \sigma_{V_{CM}} \sqrt{\gamma \pi^{CRRA,*'} \bar{\mu}} \quad \textit{where} \quad \pi^{CRRA,*} = \frac{\bar{\mu}' \Omega^{-1}}{\gamma} \tag{6.12}$$

(ii)in case of borrowing constraints

$$\mu_{V_{CM}} = r + \frac{\gamma_{inter} + \sqrt{\gamma_{inter}^2 - \hat{\mu} S - \gamma_{inter} S \sigma_{V_{CM}}^2 + \hat{\mu} S^2 \sigma_{V_{CM}}^2}}{S} \tag{6.13}$$

$$\textit{where} \quad \hat{\mu} = \gamma \pi^{CRRA,*'} \bar{\mu} \quad \gamma_{inter} = \bar{\mu}' \Omega^{-1} \mathbf{1} \quad S = \mathbf{1}' \Omega^{-1} \mathbf{1} \tag{6.14}$$

In particular, the risk aversion level for which the borrowing constraint is binding is defined by γ_{inter} and $\bar{\mu}' = (\mu_1 - r, ..., \mu_n - r)$ is the vector of excess returns and Ω denotes the variance covariance matrix.[7]

Proof. For the solution to (ii) it is enough to solve

$$\max_{\pi} \pi' \bar{\mu} \quad s.t. \pi' \Omega \pi = \sigma_V \quad \pi' \mathbf{1}.$$

Lagrangian optimization leads then to the efficient frontier. A detailed proof is, for instance, provided in Cochrane (2005) pp. 83. The solution to the portfolio weights date back to Merton (1971). □

The interpretation of the solution without borrowing constraints is straightforward. Here μ^G denotes the drift of the optimal $(\mu_{VCM}, \sigma_{VCM})$ for an investor with risk aversion $\gamma = 1$. This defines the optimal portfolio held by every investor in the economy. Thus, the relation between the volatility and the square root of $\hat{\mu}$ defines the curve on which the efficient portfolio drifts are located. The interpretation of the solution under

[6] In fact, the problem reduces to classical mean-variance analysis as in Markowitz (1952).

[7] Technically Ω has to be positive semidefinit to guarantee the invertibility.

Model and strategy parameters

	Asset 1	Asset 2
μ_i	0.11	0.08
σ_i	0.3	0.15

Interest rate and correlation	
r	0.039
$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$	-0.3

Strategy parameters	
$V_0 = S_0$	1
T	10

Table 6.1 Benchmark setup and strategy parameters.

borrowing constraints is not that straightforward. Here, it is interesting to notice that the borrowing constraints become binding for $\gamma > \gamma_{inter}$. The *pseudo* increase in the risk free interest rate is exactly the difference between the optimal unrestricted portfolio weights and the portfolio weights which exactly match the borrowing constraint standardized by $\mathbf{1}'\Omega^{-1}\mathbf{1}$.[8]

We end this section by an illustration of the efficient curve on which the optimal solution for an agent with risk aversion γ without and with borrowing constraints is located. The benchmark parameter constellation is summarized in Table 6.1 which serve as the base case throughout this chapter if not stated differently. Figure 6.1 depicts the efficiency curves dependent on the portfolio volatility $\sigma_{V_{CM}}$. The exact point on the line which is chosen by the investor only differs with respect to his risk aversion and thus determines the optimal portfolio combination $(\mu^*_{VCM}, \sigma^*_{VCM})$. Notice that the intersection point for which the borrowing constraints becomes binding corresponds to an investor with risk aversion $\gamma_{inter} = 3.69$. Thus, every investor being less risk averse will suffer from the imposed borrowing constraints.

6.2.2 HARA Utility

A HARA investor, in contrast, is characterized by an utility function of the form

$$u^{\mathrm{HARA}}(x) = \frac{(x - \tilde{G})^{1-\gamma}}{1-\gamma}$$

[8] For a detailed interpretation of the inverse of the Variance Covariance Matrix we refer to Stevens (1998).

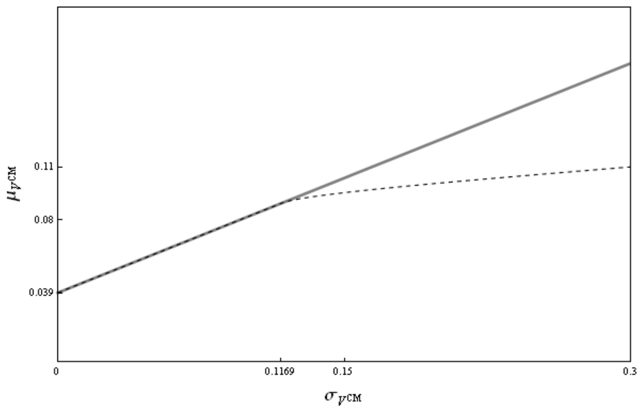


Fig. 6.1 Efficient $(\mu_{VCM}, \sigma_{VCM})$ tuples with and without borrowing constraints
The figure shows the efficient $(\mu_{VCM}, \sigma_{VCM})$ tuples without (solid line) and with borrowing constraints (dashed line).

where $\tilde{G}$ is to be interpreted as a subsistence level of the investor. Notice that there is a close link between the HARA investor and a CRRA investor. The optimization problem of a HARA investor can be reduced to a CRRA investor who maximizes the expected difference of terminal wealth and $\tilde{G}$, i.e.

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}\left[u^{\mathrm{HARA}}(V_T)\right] &= \mathbb{E}_{\mathbb{P}}\left[u^{\mathrm{CRRA}}(V_T - \tilde{G})\right] \\ &= \mathbb{E}_{\mathbb{P}}\left[u^{\mathrm{CRRA}}(C_T)\right] \\ \text{where } C_t &:= V_t - e^{-r(T-t)}\tilde{G}. \end{aligned}$$

The optimal solution is then

$$\pi^{\mathrm{HARA},*} = \frac{C_t}{V_t}\pi^{\mathrm{CRRA},*} \leq \pi^{\mathrm{CRRA},*}.$$

The interpretation is straightforward as it can be viewed in terms of a CPPI, see Basak (2002). For a given subsistence level $\tilde{G}$ the optimal solution is a constant mix strategy on the cushion C_t where $\pi^{\mathrm{CRRA},*}$ are the so-called optimal multiplier of the CPPI. In general, the idea is to invest a multiple of the cushion $C_t := V_t - e^{-r(T-t)}G$ in the risky assets. In particular, if $\tilde{G}$ is identical to the terminal wealth guarantee G, then a CPPI meets the guarantee for sure. Thus, the guarantee does not have to be backed up by an additional put option. Stated differently, the CPPI can be interpreted as a buy and hold strategy of a constant mix strategy with an additional investment into G zero-coupon bonds, i.e.[9]

$$V_t^{CPPI} = e^{-r(T-t)}G + \frac{C_0}{V_0}V_t^{CM}. \tag{6.15}$$

But a CPPI in the classical sense is only achieved for portfolio weights (multiplier) above one which can imply portfolio weights above one. In practice, however, a retail investor is restricted by short–selling and borrowing constraints. To get the intuition consider the situation of one risky asset. Then, a natural candidate for the optimal portfolio weight under borrowing constraints is given by

$$\pi_t^{HARA,*,BC} = \min\left\{\frac{\mu - r}{\gamma\sigma^2} \quad \frac{C_t}{V_t}, 1\right\}. \tag{6.16}$$

To put it in words, the investor chooses the minimum of what he would invest without borrowing constraints (the classical CPPI) and the maximum level of investment with the leverage constraint, i.e. a buy and hold in the risky asset. However, Grossman and Villa (1992) argue that this is a myopic strategy. The investor behaves as if the borrowing constraints do not exist. However, this does not necessarily be optimal. In particular, staying in the same strategy class but generalizing the natural candidate by scaling the risk aversion γ with some constant $a > 0$ cannot make the investor worse off. In fact, Grossman and Villa (1992) show that this is the solution to the infinite horizon problem, i.e. $T \to \infty$. Here, the investor's implicit risk aversion is smaller (larger) than γ for $\gamma > 1$ ($\gamma < 1$) which implies that he invests more (less) in the risky asset. Intuitively, it is clear that for very high γ deviating to a riskier strategy than the optimal unrestricted solution can be very costly (very concave utility function). In contrast, for a moderate risk averse agent the chance to offset the loss in utility due to the borrowing constraints

[9] For a detailed discussion of this representation we refer to Balder and Mahayni (2010).

can result in a riskier strategy.[10] In the following, we take the infinite horizon solution as a heuristic for the finite horizon problem where no closed-form solution exists.[11]

Remark 6.1. In the case of borrowing constraints, the investor chooses the risky portfolio weight as follows:

$$\pi_t^{HARA,*,BC} = \min\left\{\frac{\mu - r}{a\gamma\sigma^2} \quad \frac{C_t}{V_t}, 1\right\}, \tag{6.17}$$

where $\frac{1}{a\gamma} = \frac{1}{\bar{\gamma}}$.

Thus, for our risk averse investors with $\gamma > 1$ we expect a higher multiplier to start with in the case of borrowing constraints compared to the optimal multiplier of the classical CPPI case. In a multi asset setting the question arises what happens if the borrowing constraints are binding. We deal with a risk averse investor who wants a diversified portfolio. Thus, we assume that the multiplier for the risky assets for binding borrowing constraints are:

Remark 6.2. The portfolio weights (multiplier) for which the borrowing constraints are exactly met are defined by:

$$\pi_{inter}^{*,CRRA} = \frac{\gamma\bar{\mu}'\Omega^{-1}}{\mathbf{1}'\pi^{*,CRRA}}. \tag{6.18}$$

This, in fact, results in a portfolio allocation of an investor with $\gamma = 3.69$ for the benchmark parameter constellation. Then, the optimal portfolio weights for a capped CPPI on n-risky assets can be defined as follows:

Proposition 6.3 (Optimal portfolio weights capped CPPI). *The portfolio weights for the capped CPPI (CCP) are defined by*

$$\pi_t^{CCP} = \min\left\{\frac{1}{a}\pi^{CRRA,*}\frac{C_t}{V_t}, \pi_{inter}^{CRRA,*}\right\} \tag{6.19}$$

[10] Notice that the introduction of borrowing constraints result in a path-dependent strategy which leads to a loss in utility for a Black Scholes Model. For this two effects depending on γ see also Grossman and Villa (1992).

[11] The approximation is the better the longer the time to maturity of the contract.

where $\pi^{CRRA,*}$ *denote the optimal CRRA weights adjusted by a constant* $a > 0$ *and* $\pi^{CRRA,*}_{inter}$ *are the portfolio weights for which the borrowing constraints are exactly met.*

Proof. The proof is an immediate consequence of Remark 6.1 and Remark 6.2. □

For the comparison of CPPI and protective put we have to rely on one utility function. Therefore, we restrict ourselves to CRRA utility and calculate utility losses.[12] Notice that this implies that the optimal portfolio weights for the CPPI and CCP under a HARA utility function are not the optimal portfolio weights under a CRRA utility function. However, the *weights (multiplier)*, i.e. the drift and volatility of the constant mix strategy on the CPPI cushion must again be located on the efficient set defined in Proposition 6.2.

An illustration of the impact of CRRA utility on the multipliers of the constant mix on the cushion for simple CPPI and CCP is given in Table 6.2. Notice that for the CPPI the strategy becomes riskier for higher risk aversion levels whereas the CCP reaches its maximum risk for intermediate risk aversion levels. Recall that for the CCP the implied risk aversion depends on γ which defines how the optimal unrestricted solution is modified to $\tilde{\gamma}$.

6.3 Utility loss

To assess the utility losses due to the terminal wealth guarantee $WG^*_T(G,\alpha^*)$, the borrowing constraints $WG^*_T(G,\alpha^*,BC)$, the use of a CPPI (CPPI,CCP) and the *mispriced* put option $WG^*_T(G,\tilde{\alpha},BC)$ we restrict the analysis to a CRRA investor. The comparison is based on the certainty equivalents (CE) at time T. This certainty equivalent is the deterministic amount for which the investor is indifferent between getting the deterministic amount at T or using the different investment strategies with terminal wealth guarantee. For a CRRA investor who does not face any restrictions, the optimal solution is given by a CM strategy with investment fraction $\pi^{*,CRRA}$, see Merton

[12] With respect to the validity of CRRA utility and the parameter of risk aversion we refer to Munk (2008) and the literature overview therein.

γ	implied $\tilde{\gamma}$ CPPI-CRRA	implied $\tilde{\gamma}$ CCP-CRRA	difference multipliers CPPI %-increase in π	difference multipliers CCP %-increase in π
1.2	0.74	0.44	63%	172%
1.6	0.86	0.48	86%	234%
2.0	0.96	0.52	107%	281%
2.4	1.07	0.56	125%	329%
2.8	1.16	0.62	140%	351%
3.2	1.26	0.69	155%	357%
3.6	1.35	0.84	166%	329%
4.0	1.44	1.05	178%	282%

Table 6.2 Risk aversion vs. implied risk aversion
The Table compares the risk aversion of the investor with the implied risk aversion of a CPPI and a CCP under CRRA utility.

(1971). The corresponding maximal certainty equivalent CE_T^*, which will serve as the benchmark in the subsequent analysis, is given by

$$CE_T(\pi^{CRRA}) = V_0 e^{\left(\mu_{VCM}^* - \frac{1}{2}\gamma\sigma_{VCM}^{2*}\right)T}. \tag{6.20}$$

The utility loss is then measured by the so-called loss rate l which gives the annualized loss in the certainty equivalent due to the use of the investment strategies with terminal wealth guarantee.

Definition 6.2 (Loss rate). The loss rate l_T^B with B in $WG_T^*(G,\alpha^*)$, $WG_T^*(G,\alpha^*,BC)$, CPPI, CCP or $WG_T^*(G,\tilde{\alpha},BC)$ relative to the optimal solution CE_T^* over an investment horizon of T years is

$$l_T^B = \frac{\ln \frac{CE_T^*}{CE_T(B)}}{T}. \tag{6.21}$$

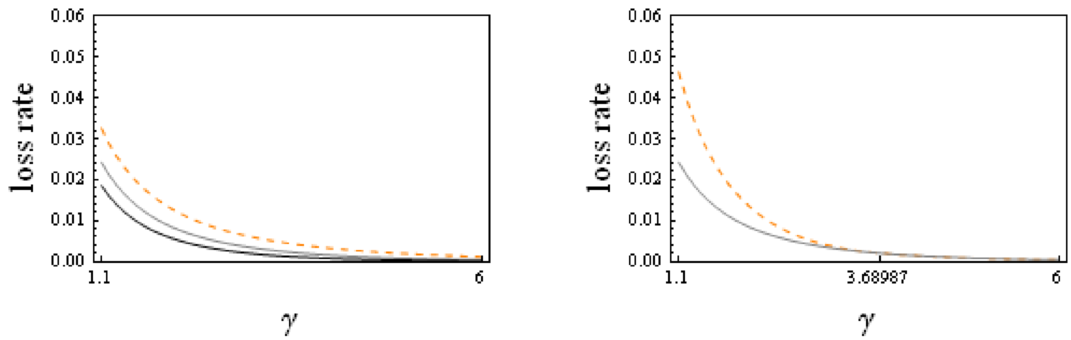


Fig. 6.2 Utility loss caused by guarantee and borrowing constraints for the protective put case. The left picture shows the impact of guarantee rates ($g = 0$ (black solid), $g = 0.01$ benchmark case (gray solid), $g = 0.02$ (gray dashed)) on the utility loss of $WG_T^*(G, \alpha^*)$ for varying γ. The right picture shows the impact of borrowing constraints on the utility loss for varying γ. The gray solid line shows the benchmark case without borrowing constraints whereas the gray dashed line is with borrowing constraints, i.e. $WG_T^*(G, \alpha^*, BC)$.

6.3.1 Utility loss caused by the guarantee and borrowing constraints

Recall that for $G = 0$ and no borrowing constraints the loss rate is equal to zero. Here, the optimal solution coincides with the unrestricted Merton solution. The left picture of Figure 6.2 depicts the loss in utility for different guarantee values dependent on the risk aversion γ. In line with intuition, the higher the guarantee the higher the loss in utility (from 1.8% for $g = 0$ to 3.2% for $g = 0.02$). For very risk averse investors the guarantee causes almost no utility loss. In this case, the investor conducts a very conservative strategy with a small proportion invested in the risky assets, thus the value of the put option is rather low, i.e. she almost looses no risk capital due to the guarantee. As already suggested the borrowing constraints harm investors with low risk aversion much more, see the right picture of Figure 6.2. For $\gamma = 1.1$ the utility loss is even doubled (from 2.4% to 4.8%). As shown in Figure 6.1 for $\gamma = 3.69$ the borrowing constraints are no longer binding. To summarize, whereas very risk averse investors neither suffer that much from guarantees nor from borrowing constraints the opposite is true for less risk averse investors.

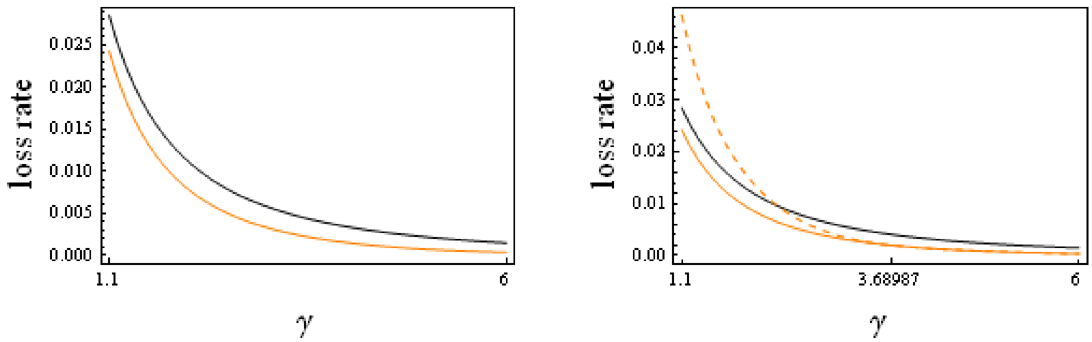


Fig. 6.3 Utility loss caused by CPPI strategy.
The left picture gives the utility loss caused by the use of a CPPI strategy (black solid line) instead of the optimal put solution (gray solid line) for varying risk aversion. The right picture compares the CPPI strategy with the put solution under borrowing constraints $WG_T^*(G,\alpha^*,BC)$(gray dashed line). In addition, the optimal put solution $WG_T^*(G,\alpha^*)$ is depicted as a benchmark.

6.3.2 Utility loss caused by CPPI

For the comparison of CPPI and protective put, it is important to keep in mind that $\pi^{*,CRRA}$ is the optimal strategy parameter for the protective put with $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$ and for the CPPI with $u(x) = \frac{(x-G)^{1-\gamma}}{1-\gamma}$. Comparing the protective put and the CPPI with adjusted strategy parameter under a CRRA utility still favors the CRRA optimal strategy, i.e. the protective put. However, the loss due to taking a CPPI instead is almost to be neglected with 0.3%. This is illustrated in the left picture of Figure 6.3 for varying γ. The right picture compares the CPPI under CRRA utility with the protective put under borrowing constraints. An investor with very low risk aversion is suffering more from reducing the portfolio weights to meet the borrowing constraints than buying too many zero bonds (she can gain about 2.2%). But already intermediate risk averse investors prefer the loss in utility by the CRRA optimal protective put under borrowing constraints to the use of the CPPI. The utility loss, however, remains very low for choosing the CPPI instead of the protective put under borrowing constraints. In a second step, we want to compare the path-dependent capped CPPI (CCP) with the CPPI and the protective put under borrowing constraints, see Figure 6.4.

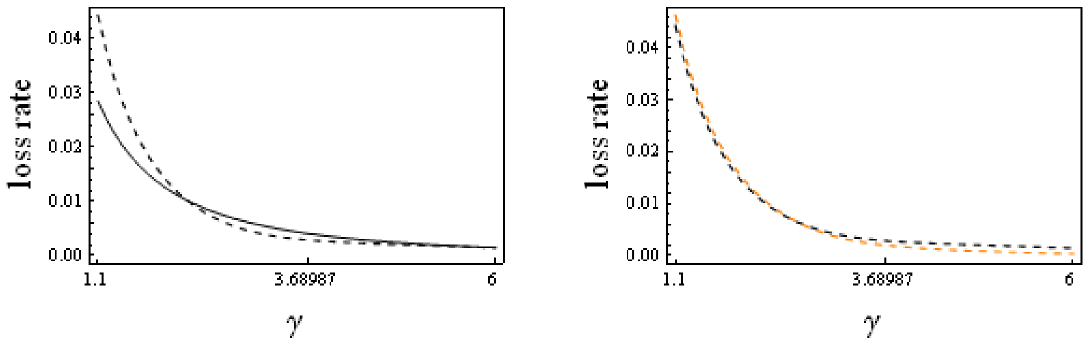


Fig. 6.4 Utility loss caused by capped CPPI (CCP) strategy.
The left picture compares the utility loss of the CPPI (black solid line) and the CCP (black dashed line) for varying γ. The right picture shows the CCP and the protective put strategy under borrowing constraints $WG_T^*(G, \alpha^*, BC)$ for varying γ.

Interestingly, although the CCP is not utility maximizing for a HARA investor, for intermediate risk aversion the CCP slightly outperforms the classical CPPI under CRRA utility. It is worth to emphasize that there is almost no difference in the loss rate between CCP and protective put strategy under borrowing constraints. The loss rates are almost identical for binding borrowing constraints and the put strategy dominates as soon as the portfolio weights of the optimal strategy are identical to the unrestricted solution.

6.3.3 Mispriced put options vs. suboptimal investment strategies

The price of the put option is deducted from the initial investment of the investor and serves as the amount the issuer needs to hedge the guarantee. Therefore, the pricing of the put option determines the fee for the guarantee. In our model setup the only risk parameter which determines the price of the put option, is given by the constant volatility.[13] There exist various reasons why the volatility used for pricing the put option differs from the volatility of the expected utility maximizing strategy of the investor. Just to mention a few: conservative volatility estimates, implied volatility instead of

[13] For pricing the put option we rely on the well-known Black-Scholes formula.

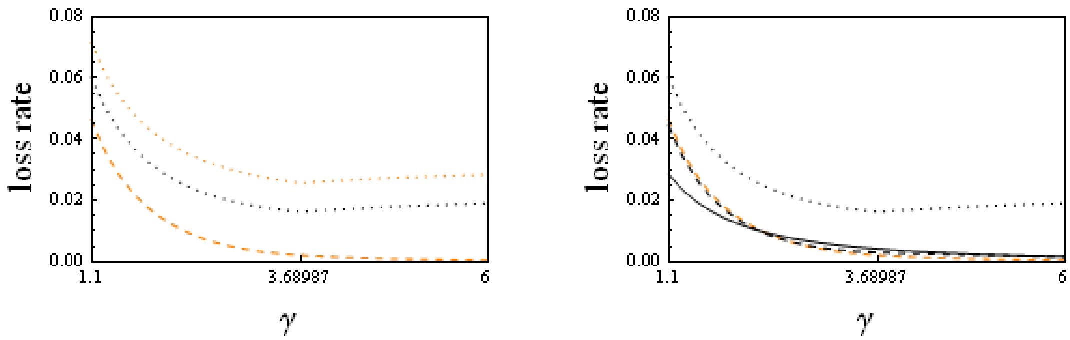


Fig. 6.5 Utility loss caused by mispriced put option.
The left picture compares $WG_T^*(G,\alpha^*,BC)$ (gray dashed)with $WG_T^*(G,\tilde{\alpha}=0.8,BC)$ (black dotted) and $WG_T^*(G,\tilde{\alpha}=0.7,BC)$ (gray dotted). The right picture depicts the $\tilde{\alpha}=0.8$ strategy with the optimal fair solution, the CPPI (black solid) and the CCP (black dashed).

historical volatility etc. In particular, it is straightforward to assume that the volatility for pricing the put option exceeds the volatility implied by the optimal payoff. Figure 6.5 shows the impact of a higher reduction in the initial investment on the utility loss. The left picture compares the utility loss of the optimal protective put where the initial investment is below the *fair* one. The misspecified $\tilde{\alpha}$-values are set to 70% and 80%. Obviously, the loss in utility is quite substantial (1.5%-3%) and well exceeds the loss due to the guarantee and the borrowing constraints. Notice that in this situation the investor suffers twice. On the one hand, the participation in the investment strategy decreases from α^* to $\tilde{\alpha}$, i.e. the slope decreases. On the other hand, the investor starts to participate later on in the gains of the underlying investment strategy, i.e. the put comes later out of the money $\frac{G}{\tilde{\alpha}} > \frac{G}{\alpha^*}$. The CPPI and the CCP with CRRA parameters both perform better than $WG_T^*(G,0.8,BC)$. The $\tilde{\alpha}$ value for which the CCP and the CPPI are dominating is given by 0.92.

6.4 Conclusion

The main focus of this chapter is on the comparison of protective put and CPPI strategies under borrowing constraints. For the comparison we rely on expected utility theory and consider a CRRA investor. Here, the *simple* CPPI can never be utility maximizing. However, the introduction of borrowing constraints shows that the capped CPPI can outperform a protective put under borrowing constraints. This gives hints why the CPPI is preferred in practice. In addition, we argue that the price for the put options bought at inception can exceed the price implied by the optimal fair contract, i.e. the price for the guarantee does not need to coincide with the endogenously determined price for the guarantee of the optimal investment strategy underlying the protective put payoff. This leads to an substantial loss in utility and the CPPI clearly dominates the protective put.

Chapter 7

Conclusion and Future Research

This thesis tries to approach the question whether recently offered structured life insurance and investment products fulfill the retail investors' needs and expectations towards an investment. The thesis is composed of two parts where the first considers structured life insurance products and the second part is dedicated to the analysis of structured investment products. Part I starts with a literature overview and points out interesting research questions still unsolved. Here, the two following chapters try to analyze some of the questions which are posed. Chapter 3 analyzes which fair combination of a self-financing investment strategy and a guarantee scheme (variable annuity or equity-linked) is the optimal choice for an insured who maximizes expected utility. For each combination of an investment strategy and a guarantee scheme, we determine the fair (arbitrage–free) combinations of guaranteed rate and participation rate. We show that a constant mix strategy combined with the structure of a variable annuity is optimal for a CRRA investor. An investor who has a subsistence level optimally chooses a CPPI strategy which meets the guarantee by construction. We point out that it is crucial to decide on guarantee scheme and investment strategy simultaneously. All results are quantified by numerical examples. It turns out that the difference between the guarantee schemes can basically be ignored. However, the loss rates for the CPPI strategy can be significant. For future research it can be interesting to consider the problem also for an entire cohort of insureds and to extend the analysis to periodic premia where prefinancing of the premiums is not feasible. Chapter 4 considers an additional option recently offered to the buyer of a variable annuity with guaranteed accumulations. The receiver of this option has the right to decide on the investment strategy in several risky assets and switch continuously in between. The rider gives rise to an unwanted incentive to conduct a riskier strategy. The insurance company must rely on a worst case strategy to hedge the guarantee for sure. The insured pays this amount at inception of the contract. To offset the loss in risk capital he then conducts a riskier strategy than the one usually implied by his risk aversion. This result is illustrated for a CRRA investor who, in addition, obtains utility from non–market wealth. The impact is quantified by means of utility losses. The main focus of this chapter, namely the incentive to conduct a riskier strategy, is true as long as the price of the protective put is deducted before the investment strategy is fixed. Thus, this result is not limited to variable annuities but is also interesting in the context of the too big to fail debate.

Further interesting research questions remain which are challenging to answer. In particular, the design of structured life insurance products has to be found which simultaneously solves the challenging risk management of these products and fulfills the expectations of the retail investor. This is left for future research.

The second part of this thesis starts with the analysis of a currently traded certificate. The relax certificate can be decomposed into a knock-out coupon bond and a knock-in minimum claim on the underlying stocks. In general, no closed-form solution exists for the price of the relax certificate. Here, analytical upper bounds for the price of the complex certificate are provided. These are compared with market prices where it turns out that the currently traded products are significantly overpriced. Thus, the question which presents itself is why these products are actually bought. This is left for future research. Chapter 6 compares two prominent examples of portfolio insurance strategies under borrowing constraints. The option based portfolio insurance (OBPI) and the constant proportional portfolio insurance (CPPI) can be justified by the solution of expected utility optimization problems. We compare the OBPI and the CPPI relying on CRRA utility. Although, the CPPI is not the optimal strategy, the introduction of borrowing constraints is enough so that the loss in utility becomes negligible. Additionally, for the protective put strategy, the price for the guarantee, i.e. the price of the option, is deducted from the initial investment premium the investor pays at inception of the contract. If the price of the put option does not correspond to the price of the put option which is implied by the optimal underlying investment strategy of the optimal payoff, the investor suffers. Here the CPPI outperforms the OBPI. An interesting point is to analyze dynamic CPPI and OBPI under borrowing constraints and transaction costs.

Appendix A

Appendix

A.1 Appendix Chapter 4

A.1.1 Proof of Theorem 4.1

First, consider part (i) of the theorem. Consider the (modified) optimization problem

$$\begin{aligned} \tilde{V}_T^{*,\text{feasible}}(\alpha,b) &:= \text{argmax}_V E_{\mathbb{P}}\left[u\left(X_T^B + V_T + [G_T - V_T]\right)^+\right] \\ \text{s.t. } E_{\mathbb{P}^*}\left[e^{-rT}V_T\right] &= \alpha \text{ and } E_{\mathbb{P}^*}\left[e^{-rT}[G_T - V_T]^+\right] \le b. \end{aligned} \tag{A.1}$$

which includes an additional (budget) restriction b compared to (4.14). Obviously, the optimal utility of the problem with the additional constraint is lower than the expected utility without the additional constraint, i.e.

$$EU\left(\tilde{V}_T^{*,\text{feasible}}(\alpha,b)\right) \le EU\left(V_T^{*,\text{feasible}}\right)$$

where we use the abbreviation $EU(V_T)$ for $E_{\mathbb{P}}\left[u\left(X_T^B + \max\{G_T, V_T\}\right)\right]$. Notice that for $0 < b_1 \le b_2$ and $0 \le \alpha_1 \le \alpha_2$ it holds (for all $\alpha > 0$ and $b > 0$)

$$\begin{aligned} EU\left(\tilde{h}_T^{*,\text{feasible}}(\alpha,b_1)\right) &\le EU\left(\tilde{V}_T^{*,\text{feasible}}(\alpha,b_2)\right), \\ EU\left(\tilde{h}_T^{*,\text{feasible}}(\alpha_1,b)\right) &\le EU\left(\tilde{V}_T^{*,\text{feasible}}(\alpha_2,b)\right). \end{aligned}$$

The above simply states that the optimal utility is increasing in the levels α and b. Let $\alpha^{*,fair} := E_{\mathbb{P}^*}\left[e^{-rT}V_T^{*,\text{fair}}\right]$ and $1-\alpha^{*,fair} := E_{\mathbb{P}^*}\left[e^{-rT}[G_T - V_T^{*,\text{fair}}]^+\right]$. For $b_2 \le b_1 = 1-\alpha^{*,fair}$ it follows

$$EU\left(\tilde{V}_T^{*,\text{feasible}}(\alpha^{\text{wc}}, b_2)\right) \leq EU\left(\tilde{V}_T^{*,\text{feasible}}(\alpha^{\text{wc}}, b_1)\right) \leq EU\left(V_T^{*,\text{feasible}}\right).$$

Finally consider a random variable $\hat{V}_T$ with $E_{\mathbb{P}^*}\left[e^{-rT}\hat{V}_T\right] = \alpha^{\text{wc}}$ and $E_{\mathbb{P}^*}\left[e^{-rT}[G_T - \hat{V}_T]^+\right] = b_2 \leq b_1$. Obviously, we have

$$EU\left(\hat{V}_T\right) \leq EU\left(\tilde{V}_T^{*,\text{feasible}}(\alpha^{\text{wc}}, b_2)\right)$$

such that $\hat{V}_T$ can not be optimal.

Part (ii) is straightforward. With respect to the set of admissible investment strategies, it holds $\alpha^{\text{wc}} = \alpha(g, \bar{\sigma}) < \alpha^{*,\text{fair}}$ and $1 - \alpha^{\text{wc}} \geq E_{\mathbb{P}^*}\left[e^{-rT}[G_T - V_T^{*,\text{feasible}}]^+\right] \geq 1 - \alpha^{*,\text{fair}}$. It immediately follows

$$EU\left(V_T^{*,\text{feasible}}\right) = EU\left(V_T^{*,\text{feasible}}(\alpha^{\text{wc}}, 1 - \alpha^{\text{wc}})\right) \leq EU\left(V_T^{*,\text{fair}}\right).$$

A.1.2 Proof of Proposition 4.2

The results are well known. For the sake of completeness, we sketch the proof. For (i), it is enough to observe that the dynamics of C are given by

$$\begin{aligned} dC_t &= C_t\left(\left(1 - \sum_{i=1}^n m_i\right) r\,dt + \sum_{i=1}^n m_i \frac{dS_{t,i}}{S_{t,i}}\right) \\ &= C_t\left(\left(1 - \sum_{i=1}^n m_i\right) r\,dt + \sum_{i=1}^n m_i\left(\mu_i\,dt + \sum_{j=1}^n b_{ij}\,dW_{t,j}\right)\right) \end{aligned}$$

where $\sum_{i=1}^n m_i \sum_{j=1}^n b_{ij} = \sqrt{\sum_{i=1}^n \sum_{j=1}^n m_i m_j \sigma_{ij}}$. In particular, reducing the dimensionality, to a one-dimensional standard $\mathbb{P}$-Brownian motion $\tilde{W}$ gives

$$C_T = C_0 e^{\left(\mu_m - \frac{1}{2}\sigma_m^2\right)T + \sigma_m \tilde{W}_T}$$

where μ_m and σ_m are defined as in Proposition 4.2. To show (ii), notice that

$$E_{\mathbb{P}}\left[u(\bar{x} + V_T)\right] = E_{\mathbb{P}}\left[\frac{(V_T + \bar{x})^{1-\gamma}}{1-\gamma}\right] = E_{\mathbb{P}}[u(C_T)] \tag{A.2}$$

where $C_t := V_t + e^{-r(T-t)}\bar{x}$. It is now straightforward to show that the optimal solution is given by a constant proportion strategy (Merton strategy (M)) w.r.t. the cushion process C. Accordingly, the optimal strategy (portfolio weights, respectively) π^* w.r.t. V is (are) given by

$$\pi_0^* = 1 - \sum_{i=1}^{N} \pi_i^* \text{ and } (\pi_1^*, \ldots \pi_N^*)' = \frac{\bar{\mu}'\Omega^{-1}}{\gamma}\frac{C_t}{V_t}. \tag{A.3}$$

For $\bar{x} = 0$, (iii) matches with the result of Proposition 2.2 in El Karoui et al. (2005). However, the result is true for arbitrary utility functions, see their Proposition 5.1. To achieve the result for $\bar{x} \neq 0$ it is enough to notice that

$$E_{\mathbb{P}}\left[u(\bar{x} + V_T)\right] = E_{\mathbb{P}}\left[\frac{(V_T + \bar{x})^{1-\gamma}}{1-\gamma}\right] = E_{\mathbb{P}}[u^{HARA}(V_T)] \tag{A.4}$$

where $u^{HARA}(x) = \frac{(x+\bar{x})^{1-\gamma}}{1-\gamma}$.

A.1.3 Proof of Proposition 4.3

The optimal strategy is immediately implied by Proposition 4.2.The initial investment is determined by Equation (4.1). $C_t := V_t + e^{-r(T-t)}\bar{x}$ yields

$$E_{\mathbb{P}^*}\left[e^{-rT}\left[e^{gT} - V_T\right]^+\right] = E_{\mathbb{P}^*}\left[e^{-rT}\left[e^{gT} - (C_T - \bar{x})\right]^+\right].$$

C is lognormal with $C_0 = V_0 + e^{-rT}\bar{x} > 0$. Therefore, $\left[e^{gT} - (C_T - \bar{x})\right]^+ = 0$ a.s. for $\bar{x} \leq -e^{gT}$ such that the put value is zero for $\bar{x} \leq -e^{gT}$. Now, consider $\bar{x} > -e^{gT}$. W.r.t. $\mathbb{P}^*$, the dynamics of C are

$$dC_t = C_t\left(r\,dt + \sigma_m\,dW_t^*\right) \text{where } \sigma_m^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} m_i m_j \sigma_{ij}.$$

It follows

$$\begin{aligned} E_{\mathbb{P}^*}\left[e^{-rT}\left[e^{gT} - (C_T - \bar{x})\right]^+\right] &= \mathscr{B}^{\text{Put}}\left(C_0, 0, r, \hat{K}, T\right) \\ = -C_0\mathcal{N}\left(-d^{(1)}\left(0, \frac{C_0}{e^{-rT}\hat{K}}\right)\right) &+ e^{-rT}\hat{K}\mathcal{N}\left(-d^{(2)}\left(0, \frac{C_0}{e^{-rT}\hat{K}}\right)\right) \end{aligned}$$

where $\hat{K} = e^{gT} + \bar{x}$ and all–in volatility $\sigma\sqrt{T} = \sigma_m\sqrt{T}$ (for $d^{(1)}$ and $d^{(2)}$). With $\bar{V}_0 = \alpha$, $C_0 = \alpha + e^{-rT}\bar{x} > 0$ and Equation (4.1)

$$\alpha = \frac{e^{-rT}\bar{x}\mathcal{N}\left(-d^{(1)}\left(0, \frac{C_0}{e^{-rT}\hat{K}}\right)\right) + 1 - (e^{(g-r)T} + e^{-rT}\bar{x})\mathcal{N}\left(-d^{(2)}\left(0, \frac{C_0}{e^{-rT}\hat{K}}\right)\right)}{\mathcal{N}\left(d^{(1)}\left(0, \frac{C_0}{e^{-rT}\hat{K}}\right)\right)}.$$

Inserting $C_0 = \alpha + e^{-rT}\bar{x} > 0$ and $\hat{K} = e^{\tilde{g}T} = e^{gT} + \bar{x}$ gives

$$\alpha = \frac{e^{-rT}\bar{x}\mathcal{N}\left(-d^{(1)}\left(0, \frac{\alpha+\bar{x}e^{-rT}}{e^{(\tilde{g}-r)T}}\right)\right) + 1 - e^{(\tilde{g}-r)T}\mathcal{N}\left(-d^{(2)}\left(0, \frac{\alpha+\bar{x}e^{-rT}}{e^{(\tilde{g}-r)T}}\right)\right)}{\mathcal{N}\left(d^{(1)}\left(0, \frac{\alpha+e^{-rT}\bar{x}}{e^{(\tilde{g}-r)T}}\right)\right)}. \quad \text{(A.5)}$$

Together, we have

$$\alpha(\tilde{g}, \sigma, \bar{x}) = 1_{\{\bar{x} \leq -e^{gT}\}} + \tilde{\alpha} 1_{\{\bar{x} > -e^{gT}\}}$$

where $\tilde{\alpha}$ satisfies condition (A.5).

A.1.4 Proof of Proposition 4.4

Notice that

$$\begin{aligned}
&E_{\mathbb{P}}\left[u(x + GMAB_T^{l,m}\right] \\
=&E_{\mathbb{P}}\left[u(x + e^{gT} + \left(V_T^{GM_{l,m}} - e^{gT}\right)^+)\right] \\
=&\frac{1}{1-\gamma}E_{\mathbb{P}}\left[(x+e^{gT})^{1-\gamma}1_{\left\{V_T^{GM_{l,m}} \leq e^{gT}\right\}} + \left(x + V_T^{GM_{l,m}}\right)^{1-\gamma}1_{\left\{V_T^{GM_{l,m}} > e^{gT}\right\}}\right].
\end{aligned}$$

Let $Y := \frac{V_T^{l,m}+l}{\alpha+e^{-rT}l}$ and recall (cf. Proposition 4.2) that $\ln Y \sim N\left((\mu_m - \frac{1}{2}\sigma_m^2)T, \sigma_m\sqrt{T}\right)$. It immediately follows

$$\begin{aligned}
E_{\mathbb{P}}\left[(x+e^{gT})^{1-\gamma}1_{\left\{V_T^{GM_{l,m}} \leq e^{gT}\right\}}\right] &= (x+e^{gT})^{1-\gamma}\mathbb{P}(V_T^{GM_{l,m}} \leq e^{gT}) \\
&= (x+e^{gT})^{1-\gamma}\mathbb{P}\left(\ln Y \leq \ln\frac{l+e^{gT}}{\alpha+e^{-rT}l}\right)
\end{aligned}$$

which gives the first part of the result. Now, notice that

$$E_{\mathbb{P}}\left[\left(x+V_T^{GM_{l,m}}\right)^{1-\gamma}1_{\left\{V_T^{GM_{l,m}}>e^{gT}\right\}}\right]=E_{\mathbb{P}}\left[\left(x-l+(\alpha+e^{-rT}l)Y\right)^{1-\gamma}1_{\left\{Y>\frac{l+e^{gT}}{\alpha+e^{-rT}l}\right\}}\right]$$

A.2 Appendix Chapter 5

A.2.1 First Hitting Time - One-Dimensional Case

To derive the distribution of the first hitting time in the one-dimensional case, we use results given in He et al. (1998). They consider the probability density and distribution function of the maximum or minimum of a one-dimensional Brownian motion with drift. Along the lines of He et al. (1998), we define

$$\underline{X}_t := \min_{0\le s\le t} X_s \;\; \overline{X}_t := \max_{0\le s\le t} X_s$$

where $X_t=\alpha t+\sigma W_t$, $t\ge 0$ and α, σ are constants. W is a Brownian motion defined on some probability space.

Proposition A.1. *Let $G(x,t;\alpha)$ and $g(y,x,t;\alpha_1)$ be defined as*

$$G(x,t;\alpha):=\mathcal{N}\left(\frac{x-\alpha t}{\sigma\sqrt{t}}\right)-e^{\frac{2\alpha x}{\sigma^2}}\mathcal{N}\left(\frac{-x-\alpha t}{\sigma\sqrt{t}}\right),$$

$$g(y,x,t;\alpha_1):=\frac{1}{\sigma\sqrt{t}}\mathcal{N}'\left(\frac{x-\alpha_1 t}{\sigma\sqrt{t}}\right)\left(1-e^{-\frac{4x^2-4x4y}{2\sigma^2 t}}\right)$$

where N denotes the cumulative distribution function of the standard normal distribution and $N'(z)$ the density of the standard normal distribution.
For $x\ge 0$, it holds

$$\mathbb{P}(\overline{X}_t\le x)=G(x,t;\alpha),\qquad f^P_{\tau_m^{(1)}}=g(y,x,t;\alpha_1)dy.$$

For $x<0$, it holds

$$\mathbb{P}(\underline{X}_t\ge x)=G(-x,t;-\alpha),\qquad \mathbb{P}(X_1(t)\in dy,\underline{X}_1(t)\ge x)=g(-y,-x,t;-\alpha_1)dy.$$

Proof. : C.f. Theorem 1 of He et al. (1998) and the proof given here. □

Corollary A.1. *Let*

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$$

where μ, σ *(*$\sigma > 0$*) are constants.* W *is a Brownian motion defined on some probability space. For the first hitting time* $\tau_m := \inf\{t \geq 0 | S_t \leq m\}$*, (*$m < S_0$*) it holds that*

$$\mathbb{P}(\tau_m \leq t) = \mathcal{N}\left(\frac{\ln \frac{m}{S_0} - \left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) + e^{2\frac{\mu - \frac{1}{2}\sigma^2}{\sigma^2}\ln \frac{m}{S_0}} \mathcal{N}\left(\frac{\ln \frac{m}{S_0} + \left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right),$$

$$f^P_{\tau_m^{(1)}} = \frac{-\ln \frac{m}{S_0}}{\sqrt{2\pi\sigma^2 t^3}} e^{-\frac{1}{2}\frac{\left(\ln \frac{m}{S_0} - (\mu - \frac{1}{2}\sigma^2)t\right)^2}{\sigma^2 t}} dt.$$

Proof. Note that

$$\tau_m := \inf\left\{t \geq 0 \,\middle|\, S_t \leq m\right\} = \inf\left\{t \geq 0 \,\middle|\, \ln \frac{S_t}{S_0} \leq \ln \frac{m}{S_0}\right\}.$$

Let X_t denote the logarithm of the normalized asset price, i.e. $X_t := \ln \frac{S_t}{S_0} = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t$ and set $\alpha = \mu - \frac{1}{2}\sigma^2$. The stopping time τ_m is related to the first hitting time of a one-dimensional Brownian motion with drift α. With $x := \ln \frac{m}{S_0} < 0$ it follows

$$\mathbb{P}(\tau_m \leq t) = \mathbb{P}(\underline{X}_t \leq x) = 1 - \mathbb{P}(\underline{X}_t \geq x).$$

According to Proposition A.1, we have

$$\begin{aligned} 1 - \mathbb{P}(\underline{X}_t \geq x) &= 1 - G(-x, t; -\alpha) = 1 - \mathcal{N}\left(\frac{-x + \alpha t}{\sigma\sqrt{t}}\right) + e^{\frac{2\alpha x}{\sigma^2}} \mathcal{N}\left(\frac{x + \alpha t}{\sigma\sqrt{t}}\right) \\ &= \mathcal{N}\left(\frac{x - \alpha t}{\sigma\sqrt{t}}\right) + e^{\frac{2\alpha x}{\sigma^2}} \mathcal{N}\left(\frac{x + \alpha t}{\sigma\sqrt{t}}\right). \end{aligned}$$

Inserting α and x gives the distribution function. To derive the density function, define

$$f(t) := \mathcal{N}\left(\frac{x - \alpha t}{\sigma\sqrt{t}}\right) + e^{\frac{2\alpha x}{\sigma^2}} \mathcal{N}\left(\frac{x + \alpha t}{\sigma\sqrt{t}}\right).$$

This implies

$$f'(t) = \mathcal{N}'\left(\frac{x-\alpha t}{\sigma\sqrt{t}}\right) \times \left(\frac{-\alpha\sigma\sqrt{t} - \frac{\sigma}{2\sqrt{t}}(x-\alpha t)}{\sigma^2 t}\right)$$
$$+ e^{\frac{2\alpha x}{\sigma^2}} \times \mathcal{N}'\left(\frac{x+\alpha t}{\sigma\sqrt{t}}\right) \times \left(\frac{\alpha\sigma\sqrt{t} - \frac{\sigma}{2\sqrt{t}}(x+\alpha t)}{\sigma^2 t}\right)$$

Using $N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, we get

$$e^{\frac{2\alpha x}{\sigma^2}} \mathcal{N}'\left(\frac{x+\alpha t}{\sigma\sqrt{t}}\right) = \frac{1}{\sqrt{2\pi}} e^{\frac{2\alpha x}{\sigma^2} - \frac{1}{2}\left(\frac{x+\alpha t}{\sigma\sqrt{t}}\right)^2} = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{\sigma^2}\left[2\alpha x - \frac{1}{2t}(x+\alpha t)^2\right]}$$
$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\alpha t)^2}{\sigma^2 t}} = \mathcal{N}'\left(\frac{x-\alpha t}{\sigma\sqrt{t}}\right).$$

Inserting this in the above equation for $f'(t)$ gives

$$f'(t) = \mathcal{N}'\left(\frac{x-\alpha t}{\sigma\sqrt{t}}\right) \times \frac{-\sigma}{2\sqrt{t}\sigma^2 t}(x - \alpha t + x + \alpha t) = \frac{-x}{\sqrt{2\pi\sigma^2 t^3}} e^{-\frac{1}{2}\frac{(x-\alpha t)^2}{\sigma^2 t}}.$$

Using $\alpha = \mu - \frac{1}{2}\sigma^2$ and $x = \ln\frac{m}{S_0}$ gives the result. □

A.2.2 First Hitting Time – Two Dimensional Case

The distribution of the first hitting time of a two-dimensional arithmetic Brownian motion is given in He et al. (1998) and Zhou (2001):

Proposition A.2. *Let $X_t^{(j)} = \alpha_j t + \sigma_j W_t^{(j)}$ ($j = 1,2$), where α_j and σ_j are constants. $W^{(1)}$, $W^{(2)}$ are two correlated Brownian motions with $\left\langle W^{(1)}, W^{(2)}\right\rangle_t = \rho t$. Then, the probability that $X^{(1)}$ and $X^{(2)}$ will not hit the upper boundaries $x^{(1)} > 0$ and $x^{(2)} > 0$ up to time t is given by*

$$\mathbb{P}^*\left(\overline{X}_t^{(1)} \leq x^{(1)}, \overline{X}_t^{(2)} \leq x^{(2)}\right) = \frac{2}{\alpha t} e^{a_1 x_1 + a_2 x_2 + bt} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right)$$
$$e^{-\frac{r_0^2}{2t}} \int_0^{\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right) g_n(\theta) d\theta$$

where $\overline{X}_t := \max_{0 \leq s \leq t} X_s$. The parameters are defined by

$$a_1 = \frac{-\alpha_1\sigma_2 + \rho\alpha_2\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2} \qquad a_2 = \frac{-\alpha_2\sigma_1 + \rho\alpha_1\sigma_2}{(1-\rho^2)\sigma_2^2\sigma_1}$$

$$d_1 = a_1\sigma_1 + a_2\sigma_2\rho \qquad d_2 = a_2\sigma_2\sqrt{1-\rho^2}$$

and by

$$b = \alpha_1 a_1 + \alpha_2 a_2 + \frac{1}{2}\sigma_1^2 a_1^2 + \frac{1}{2}\sigma_2^2 a_2^2 + \rho\sigma_1\sigma_2 a_1 a_2$$

$$\alpha = \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{if } \rho < 0 \\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{otherwise} \end{cases}$$

$$\theta_0 = \begin{cases} \tan^{-1}\left(\frac{\frac{x2}{\sigma_2}\sqrt{1-\rho^2}}{\frac{x1}{\sigma_1} - \rho\frac{x2}{\sigma_2}}\right) & \text{if } (.) > 0 \\ \pi + \tan^{-1}\left(\frac{\frac{x2}{\sigma_2}\sqrt{1-\rho^2}}{\frac{x1}{\sigma_1} - \rho\frac{x2}{\sigma_2}}\right) & \text{otherwise} \end{cases}$$

$$r_0 = \frac{x_2}{\sigma_2} / \sin(\theta_0).$$

The function g_n is defined as

$$g_n(\theta) = \int_0^\infty r e^{-\frac{r^2}{2t}} e^{d_1 r \sin(\theta-\alpha) - d_2 r \cos(\theta-\alpha)} I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{t}\right) dr.$$

$I_v(z)$ is the modified Bessel function of order v.

Proof. Cf. Proposition 1 of Zhou (2001) and the proof given there. □

Corollary A.2. *For two stocks $S^{(k)}$ and $S^{(l)}$ with volatilities σ_k and σ_l and correlation $\rho_{k,l}$, the distribution function of the first hitting time $\min\{\tau_{m,k}, \tau_{m,l}\}$ under the risk-neutral measure is given by*

$$\mathbb{P}^*\left(\min\{\tau_{m,k}, \tau_{m,l}\} \le t\right)$$
$$= 1 - \frac{2}{\alpha t} e^{a_k \ln\left(\frac{S_0^{(k)}}{m}\right) + a_l \ln\left(\frac{S_0^{(l)}}{m}\right) + bt} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \cdot e^{-\frac{r_0^2}{2t}} \int_0^\alpha \sin\left(\frac{n\pi\theta}{\alpha}\right) g_n(\theta) d\theta$$

where

$$a_k = \frac{(r-0.5\sigma_k^2)\sigma_l - \rho_{k,l}(r-0.5\sigma_l^2)\sigma_k}{(1-\rho_{k,l}^2)\sigma_k^2\sigma_l} \qquad a_l = \frac{(r-0.5\sigma_l^2)\sigma_k - \rho_{k,l}(r-0.5\sigma_k^2)\sigma_l}{(1-\rho_{k,l}^2)\sigma_l^2\sigma_k}$$

$$d_k = a_k\sigma_k + a_l\sigma_l\rho_{k,l} \qquad d_l = a_l\sigma_l\sqrt{1-\rho_{k,l}^2}$$

and

$$b = -(r-0.5\sigma_k^2)a_k - (r-0.5\sigma_l^2)a_l + \frac{1}{2}\sigma_k^2 a_k^2 + \frac{1}{2}\sigma_l^2 a_l^2 + \rho_{k,l}\sigma_k\sigma_l a_k a_l$$

$$g_n(\theta) = \int_0^\infty r e^{-\frac{r^2}{2t}} e^{d_k r\sin(\theta-\alpha) - d_l r\cos(\theta-\alpha)} I_{\frac{n\pi}{\alpha}}\left(\frac{rr_0}{t}\right) dr.$$

$I_v(z)$ is the modified Bessel function of order v. α, θ_0, and r_0 are given in Proposition A.2 in Appendix A.2.2 for the case where $k=1$ and $l=2$.

Proof. The stock prices are given by

$$S_t^{(j)} = S_0 e^{(r-\frac{1}{2}\sigma_j^2)t + \sigma_j W_t^{(j)}} \qquad j = k, l.$$

The first hitting time of the lower boundary $m_j < S_0^{(j)}$ by the geometric Brownian motion $S_t^{(j)}$ is

$$\tau_m^{(j)} := \inf\left\{t \geq 0 \,\middle|\, S_t^{(j)} \leq m_j\right\} \;=\; \inf\left\{t \geq 0 \,\middle|\, -\ln\frac{S_t^{(j)}}{S_0^{(j)}} \geq \ln\frac{S_0^{(j)}}{m_j}\right\}.$$

With the definition of the arithmetic Brownian motion

$$X_t^{(j)} := -\ln\frac{S_t^{(j)}}{S_0^{(j)}} = -\left(r - \frac{1}{2}\sigma_j^2\right)t - \sigma_j W_t^{(j)},$$

the first hitting time can be rewritten as $\tau_m^{(j)} = \inf\left\{t \geq 0 \,\middle|\, X_t^{(j)} \geq \ln\frac{S_0^{(j)}}{m_j}\right\}$. Using the relation $\{\tau_{m,j} > t\} = \left\{\overline{X}_t^{(j)} < \ln\frac{S_0^{(j)}}{m_j}\right\}$ we can conclude

$$\mathbb{P}^*\left(\min\{\tau_{m,k}, \tau_{m,l}\} > t\right) = \mathbb{P}^*\left(\overline{X}_t^{(k)} < \ln\frac{S_0^{(k)}}{m_k}, \overline{X}_t^{(l)} < \ln\frac{S_0^{(l)}}{m_l}\right).$$

Since both, $\ln\frac{S_0^{(k)}}{m_k} > 0$ and $\ln\frac{S_0^{(l)}}{m_l} > 0$, the result follows from Proposition A.2. □

A.3 Appendix Chapter 6.

A.3.1 Proof of Proposition 6.1

Let $V_0 = \alpha$ and $G = e^{gT}$. Notice that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[u(W_T(G,\alpha))\right] =& \mathbb{E}_{\mathbb{P}}\left[u(e^{gT} + \left[V_T(\phi) - e^{gT}\right]^+)\right] \\ =& \frac{1}{1-\gamma}\mathbb{E}_{\mathbb{P}}\left[e^{(1-\gamma)gT} 1_{\{V_T(\phi)\le e^{gT}\}} + (V_T(\phi))^{1-\gamma} 1_{\{V_T(\phi)>e^{gT}\}}\right].\end{aligned}$$

First, consider

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[e^{(1-\gamma)gT} 1_{\{V_T(\phi)\le e^{gT}\}}\right] &= e^{(1-\gamma)gT}\mathbb{P}\left(V_T(\phi)\le e^{gT}\right) \\ &= \mathbb{P}\left(\frac{W_T}{\sqrt{T}} \le \frac{\ln\frac{e^{gT}}{\alpha} - (\mu_{V_{CM}} - \frac{1}{2}\sigma^2_{V_{CM}})T}{\sigma_{V_{CM}}\sqrt{T}}\right).\end{aligned}$$

Now, consider

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[(V_T(\phi))^{1-\gamma} 1_{\{V_T(\phi)>e^{gT}\}}\right] &= V_0^{1-\gamma}\mathbb{E}_{\mathbb{P}}\left[e^{(1-\gamma)\left(\mu_{V_{CM}}-\frac{1}{2}\sigma_V^2\right)T+(1-\gamma)\sigma_{V_{CM}}W_T} 1_{\{V_T(\phi)>e^{gT}\}}\right] \\ &= \alpha^{1-\gamma}e^{(1-\gamma)\left(\mu_{V_{CM}}-\frac{1}{2}\gamma\sigma_V^2\right)T}\ \hat{\mathbb{P}}\left(V_T(\phi)>e^{gT}\right)\end{aligned}$$

where

$$\left(\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right)_T := e^{-\frac{1}{2}(1-\gamma)^2\sigma_V^2 T + (1-\gamma)\sigma_{V_{CM}}W_T}.$$

Notice that Girsanov's theorem implies that $\hat{W}_t = W_t - (1-\gamma)\sigma_{V_{CM}}t$ is a $\hat{\mathbb{P}}$–Brownian motion. Using

$$\begin{aligned}V_T &= V_0 e^{(\mu_{V_{CM}}-\frac{1}{2}\sigma_V^2)T+\sigma_{V_{CM}}W_T} \\ &= V_0 e^{(\mu_{V_{CM}}+(\frac{1}{2}-\gamma)\sigma_V^2)T+\sigma_{V_{CM}}\hat{W}_T}\end{aligned}$$

immediately gives

$$\hat{\mathbb{P}}\left(V_T(\phi)>e^{gT}\right) = \hat{\mathbb{P}}\left(\frac{-\hat{W}_T}{\sqrt{T}} < \frac{-\ln\frac{e^{gT}}{\alpha} + (\mu_{V_{CM}} + (\frac{1}{2}-\gamma)\sigma_V^2)T}{\sigma_{V_{CM}}\sqrt{T}}\right).$$

Appendix B
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